STOAT

SKEIN LASAGNA MODULES AND HANDLE ATTACHMENTS

AYO D&I LINDBLAD

ROUGH OUTLINE

KEY:

Only spoken

Only written

Spoken and written

Draw this,
describe
while drawing

Draw this,
don't describe
while drawing

LENGTH: ~50 min.

Hello! I'm Ayo and today I'll be telling you about handle attachments in the study of skein lasagna modules

The lovely talks these past few days have given a great introduction to skein lasagna modules, so we just need to start by introducing handles

A 4-D k-handle is a copy of $D^k \times D^{4-k}$, attached to a 4-mfld X^4 via an embedding $S^{k-1} \times D^{4-k} \hookrightarrow \partial X$

TODAY, we'll discuss how

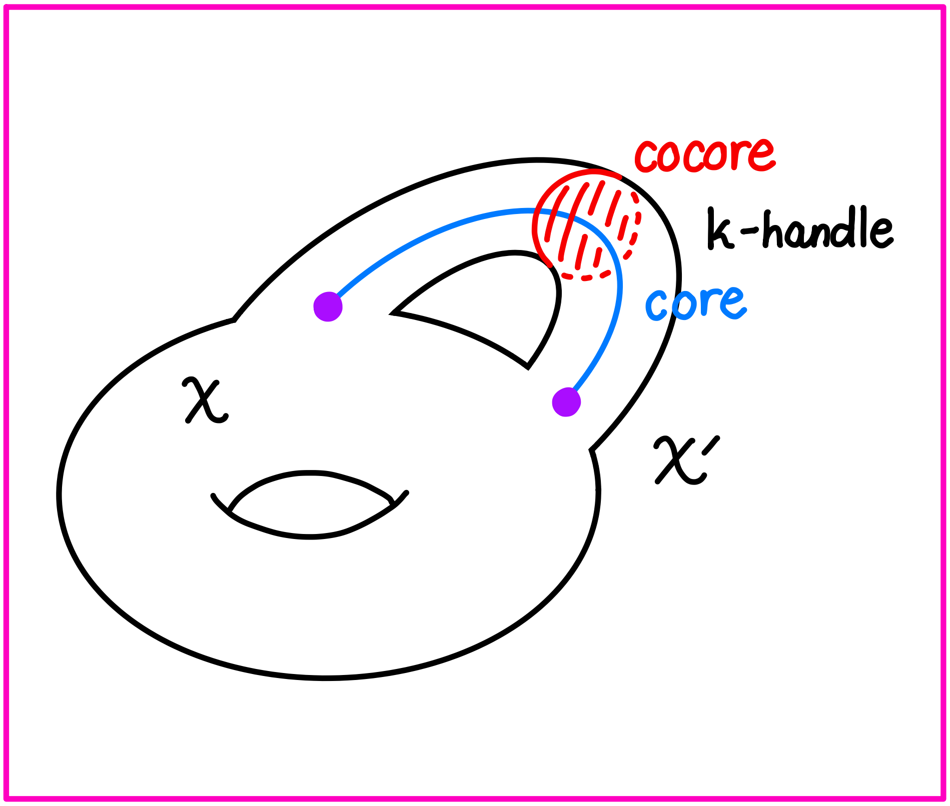
(I) $S_0^2(X) \cong S_0^2(X \cup 4\text{-handle})$ where here, we omit mention of a link in this notation to indicate that we are considering the skein lasagna module of a manifold with respect to the empty link in its boundary. Note the results we'll discuss today generalize to statements for skein lasagna modules with respect to arbitrary boundary links, but we avoid presenting this perspective for simplicity.

(II) $S_0^2(X) \twoheadrightarrow S_0^2(X \cup 3\text{-handle})$

(III) $S_0^2(B^4 \cup_L 2\text{-handle}) \cong$ a gadget we'll define called the cabled Khovanov-Rozansky homology of L $KhR_2(L)$ which is a certain quotient of an infinite direct sum of Khovanov-Rozansky homologies of cables of L. Again, we'll formally define that later.

Oh, and this is All due to Manolescu-Neithalath '22

Now, consider a 4-mfld $X' = X \cup k$-handle. Let's draw a dimensionally reduced picture:



Here, we'll mark out the core and co-core of the handle. The core then has $\boxed{\text{dim. } k}$, while the co-core has $\boxed{\text{dim. } 4-k}$.

Observe that the inclusion $i : X \hookrightarrow X'$ induces $\rightsquigarrow$ a mapping $i_* : S_0^2(X) \to S_0^2(X')$ given by simply considering a lasagna filling in X as a lasagna filling in X'.

Now, taking $k \in \{3,4\}$, we have $\Rightarrow 2 + \dim \text{cocore}$
$= 2 + (4-k) < 4$, so
by $\Rightarrow_{\pitchfork}$ any $\Sigma^2 \subset X'$ is isotopic to one in $X' \setminus \text{cocore}$, which we may observe deformation retracts onto $\cong X$.

$\Rightarrow$ any lasagna filling in X' can be isotoped to one in X

$\Rightarrow i_*$ is surj.

$\Rightarrow$ (II) is proved $\square$ $\boxed{\checkmark}$

Now, we can Consider an isotopy F between $\Sigma^2, \tilde{\Sigma}^2 \subset X'$ as a 3-submfld. of $X' \times I$.

As just noted before, we may isotope Σ and $\tilde{\Sigma}$ to only lie in X.

Then, taking $k = 4$, we have $\Rightarrow 3 + (4-k+1) < 4+1$,
so by $\Rightarrow_{\pitchfork}$ F is isotopic to a 3-submfld. of $(X' \setminus \text{cocore}) \times I$; and we can arrange that it corr. to an isotopy between $\Sigma, \tilde{\Sigma}$.

$\Rightarrow$ any surface isotopy in X' can be done in $X' \setminus \text{cocore} \cong X$.

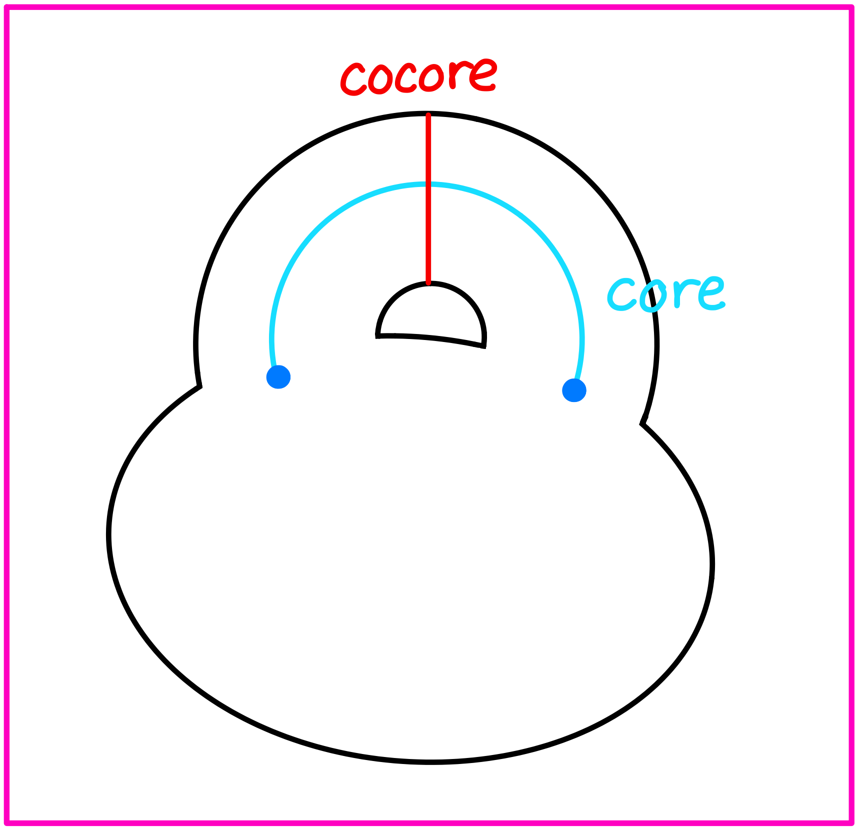
$\Rightarrow$ any isotopy of lasagna fillings in X' can be done in X

$\Rightarrow i_*$ is inj. (and thus an iso.)

$\Rightarrow$ (I) is proved $\square$ $\boxed{\checkmark}$

So, we Now just have (III) left to show. To this end, take $k=2$, $X=B^4$.

Let's draw a dimensionally reduced picture:

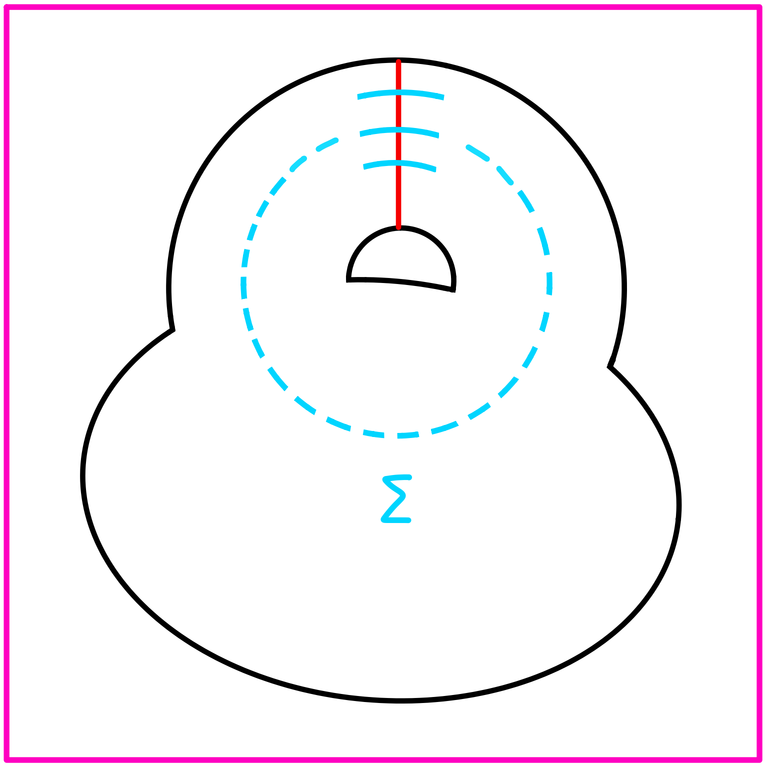


$\partial(\text{core})$ is then $=$ some knot K $\boxed{K}$ in S^3, and the pushoff of K into the core has some linking # we call n w/ K.

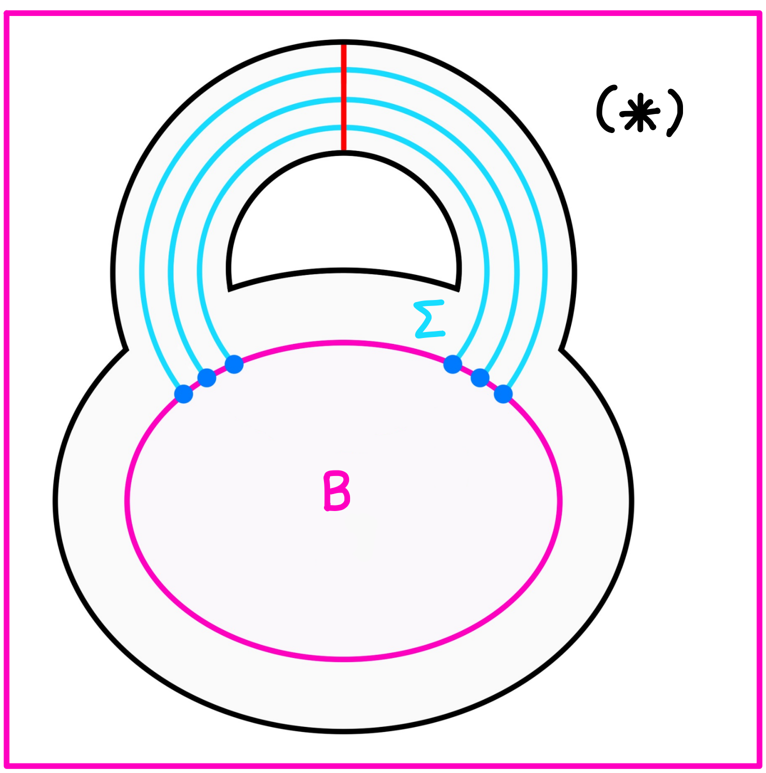
X' is then called $=:$ the n-trace of K, denoted by $X_n(K)$.

Now, let's consider a surface $\Sigma \subset X_n(K)$.

We can arrange $\Sigma \pitchfork$ cocore of the 2-handle. Near the cocore, Σ will then look like some number of parallel copies of the core; let's draw what that looks like



We can then continue to isotope Σ until it looks like parallel sheets outside a $B^4 =: B$



$\Sigma \cap \partial B$ will then be = some # k_- of − ori. copies of K
⊔ some # k_+ of + ori. copies of K

We call this link =: $K(k_-, k_+)$. As an unoriented link, this is an example of a cable of K.

Considering any lasagna filling $F := (\Sigma, \{(B_i, v_i)\})$ of $X_n(K)$ w/ surface $\Sigma \pitchfork$ cocore, input balls B_i and Khovanov-Rozansky labelings v_i,

we can then ⟶ isotope B_i to lie in B and Σ to look like (∗), looking like parallel copies of the core in the 2-handle.

Σ gives rise to ⟶ a cob. from $\cup \partial B_i \cap \Sigma$ to $K(k_-, k_+)$, and considering the induced action of this cobordism on the Khovanov-Rozansky labelings v_i then gives

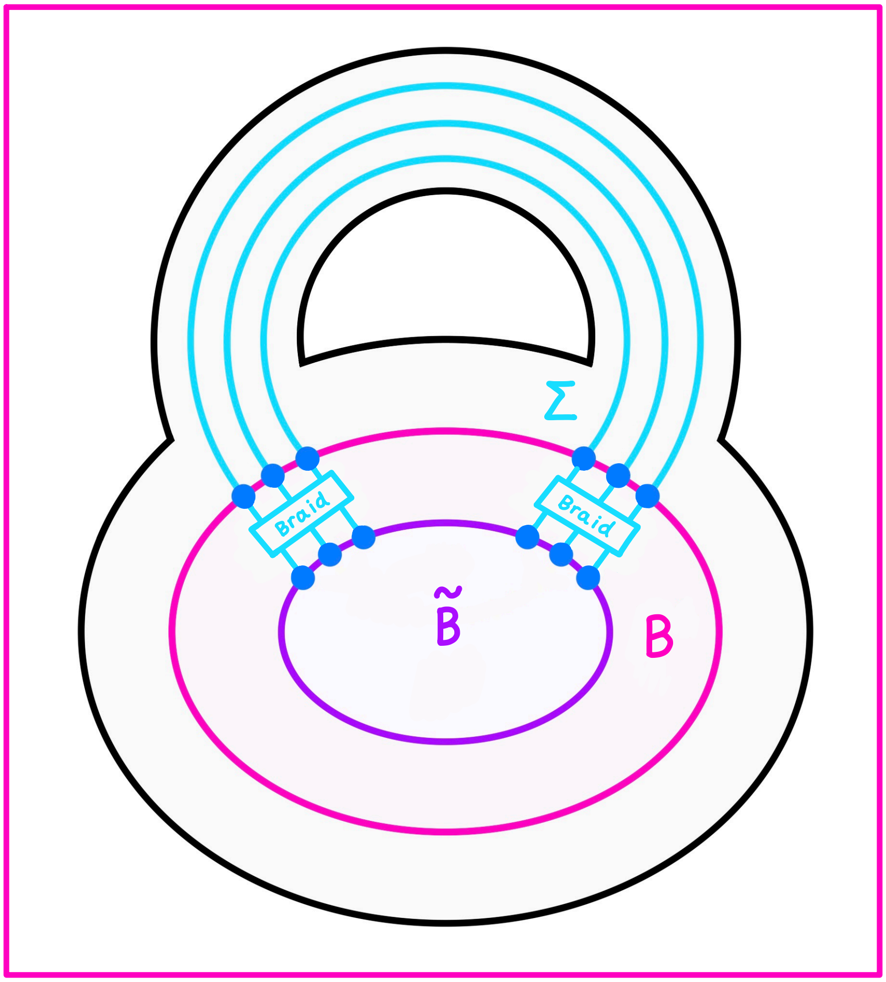
⟶ an elt. we'll call $\varphi(F)$ of $KhR_2(K(k_-, k_+))$

Now, we're on our way to getting an isomorphism between the skein lasagna module and some object built from Khovanov-Rozansky homologies of cables of K.

However, this process is Not yet well-defined on $S_0^2(X_n(K))$ - for example, we could perform an operation we call

(a) — we Could braid together sheets of Σ to change $\varphi(F)$ by the induced cob. map on $KhR_2(K(k_-, k_+))$

Let's draw what this could look like...



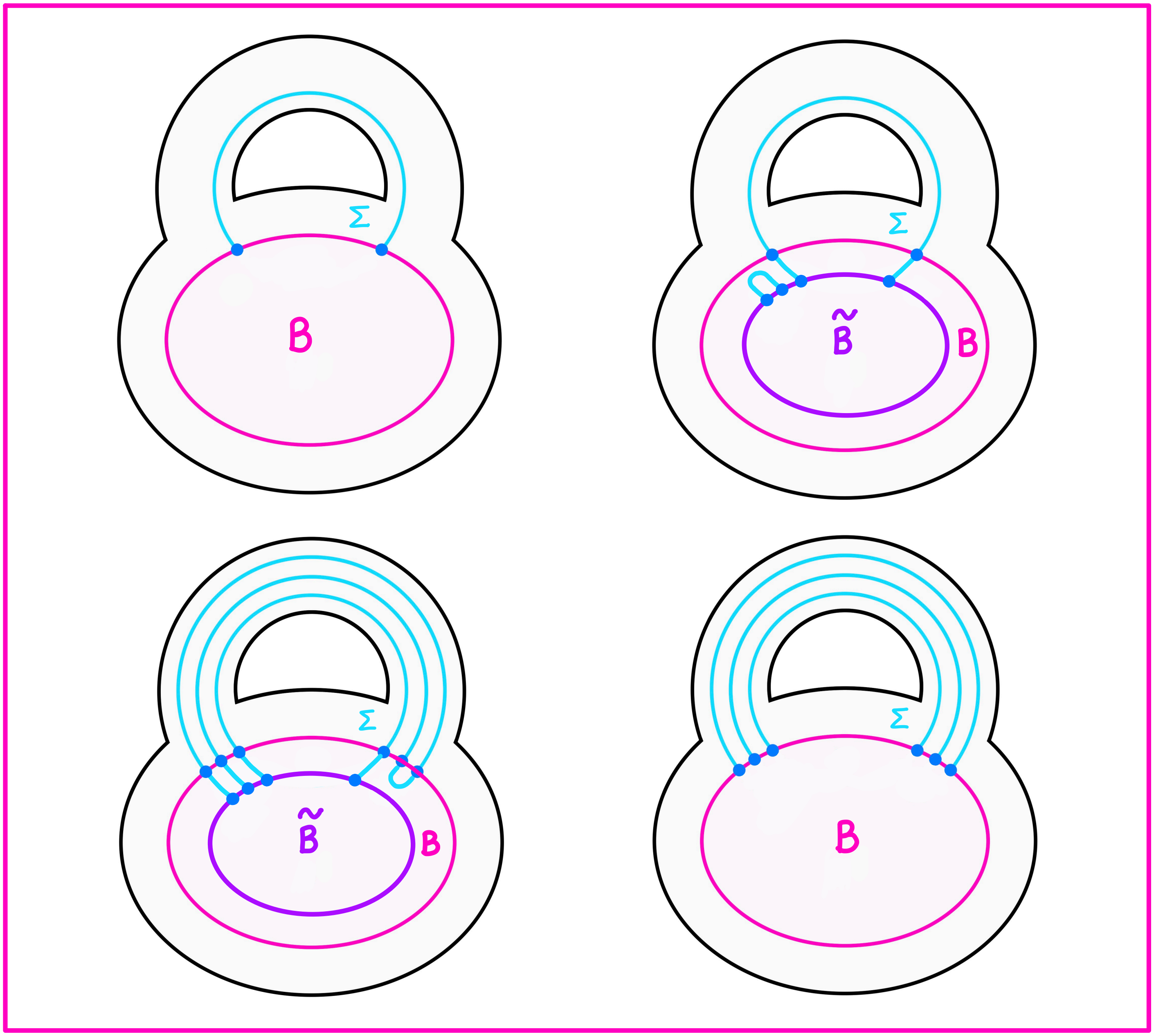
Then, $\varphi(\tilde{F}) = (\text{braid cob.})_* \varphi(F)$, so these two equivalent lasagna fillings can, under this map φ, give us different Khovanov-Rozansky labelings.

We'll denote by

(b) another way to produce such fillings showing φ is not yet well-defined on the skein lasagna module. Specifically, we can

Add a pair of opp. ori. sheets to Σ to change $\varphi(F)$ by a map $KhR_2(K(k_-, k_+)) \to KhR_2(K(k_-+1, k_++1))$

Informally, we can do this by introducing a disjoint 2-sphere to Σ in the input ball, then stretching it through the 2-handle to create 2 new parallel sheets. Let's draw this procedure:



Of course, we could also take this 2-sphere to be dotted; this would add a dot to one of the two sheets and change the way this operation alters the Khovanov-Rozansky labeling we end up with

So, we've described operations showing that our map φ we defined on lasagna fillings (whose surfaces are arranged to be transverse to the cocore) is not well-defined on equivalence classes of lasagna fillings. However, note that neither of these two operations changed the difference $k_+ - k_-$ between the numbers of positively and negatively oriented sheets: (a) keeps both the same, while (b) adds one of each.

There is a homological reason for this. Specifically,

$$k_+ - k_- = \underset{\text{of } \Sigma}{\text{+ sheets}} \pitchfork \text{core} - \underset{\text{of } \Sigma}{\text{- sheets}} \pitchfork \text{core} = [\Sigma]$$

$\in H_2(X_n(K); \partial)$ and $[\Sigma]$ is the same for any lasagna filling equiv. to F, as all surfaces of elements of this equivalence class are, by definition, homologous rel some collection of balls.

Then, writing $\alpha := k_+ - k_- = [\Sigma]$, any KhR labeling resulting from this process φ applied to $(\tilde{F} \in [F])$ lies in $\bigoplus_{r \in \mathbb{N}} KhR_2(K(r - \min(\alpha, 0), r + \max(\alpha, 0))$

Let's write $\boxed{\overset{!!}{\alpha^-}}$ and $\boxed{\overset{!!}{\alpha^+}}$ for the min and max of α and 0 respectively.

We can also see that as a map $\{$fillings w/ $[\Sigma] = \alpha\}$ $\to \bigoplus_{r \in \mathbb{N}} KhR_2(K(r - \alpha^-, r + \alpha^+))$, φ is surj. through the following:

Pick $\boxed{\overset{\Psi}{v}}$, so $\exists\ r \geq |\alpha|$ s.t. $v \in KhR_2(K(r, r + \alpha))$. Then,

Consider the lasagna filling F_v w/ input ball B slightly smaller than the 0-handle, surface Σ_v r $-$ ori sheets, $r + |\alpha|$ $+$ ori sheets (so $[\Sigma] = \alpha$), and KhR labeling v.

Then, $\varphi(F_v) = v$, so φ is surj. as a map $\boxed{\nearrow}$ from lasagna fillings with surface having relative second homology α to this infinite direct sum of Khovanov – Rozansky homologies of these cables of K. Now, say $v = \varphi(F)$ for some lasagna filling F. Running the procedure φ in reverse shows F_v is equivalent to F in the skein lasagna module.

$\Rightarrow$ can recover $[F]$ from any $v \in \varphi([F])$

$\Rightarrow$ the subsets $\varphi([F])$ are disjoint

$\Rightarrow$ they define an equiv. rel. $\sim_\varphi$ on $\bigoplus_{r \in \mathbb{N}} KhR_2(K(r-\alpha^-, r+\alpha^+))$

Therefore, Defining the (cabled KhR homology)

$$\underline{KhR}_2(K,\alpha) := \bigoplus_{r \in \mathbb{N}} KhR_2(K(r-\alpha^-, r+\alpha^+))\{-(2r+|\alpha|)\} \Big/ \sim_\varphi$$

with a grading shift we won't think too hard about now, we get an isomorphism $S_0^2(X_n(K),\alpha) \cong \underline{KhR}_2(K,\alpha)$, exactly proving our last point (III) $\square$.

Now, note that we're not quite done if we want to use this in practice because we haven't described this equivalence relation $\sim_\varphi$ very explicitly.
Manolescu and Neithalath do this formally in their paper
In specific, recall the operations (a)
and (b) we discussed which braid the sheets of a lasagna filling and add oppositely-oriented sheets. Manolescu and Neithalath discussed how the relations these induce on the infinite direct sum of Khovanov-Rozansky homologies exactly generate the desired equivalence relation whose associated quotient gives the cabled Khovanov-Rozansky homology. They also describe these induced relations very explicitly, making everything nicely computable!
But we don't have time for all that - that's all I have for today, so thanks for listening!