Boundary Dehn Twists are Often Commutators

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DIFFEOMORPHISM GROUPS

Closed oriented n-manifold X

$$X^\circ := X \setminus B^n$$

$$\mathrm{Diff}^+(X) := \left\{ \begin{array}{l} \text{orientation-preserving} \\ \text{diffeomorphisms } f : X \circlearrowleft \end{array} \right\}$$

$$\mathrm{Diff}^+(X^\circ, \partial) := \{ f \in \mathrm{Diff}^+(X^\circ) \mid f|_{\nu(\partial X^\circ)} = id \}$$

Tubular nbhd of $\partial X^\circ \cong S^3$

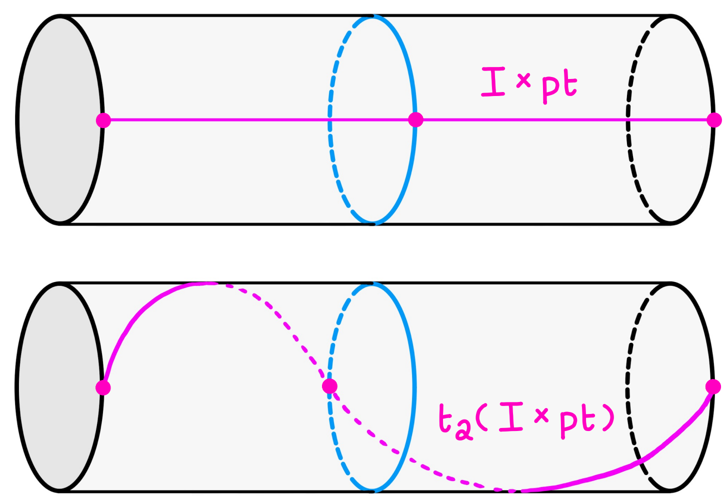
Dehn Twist t_n on $I \times S^{n-1}$

$$\underbrace{[\alpha]}_{:I := [0,1] \to SO(n)} \text{ generates } \pi_1(SO(n)) \cong \begin{cases} \mathbb{Z}; & n=2 \\ \mathbb{Z}/2\mathbb{Z}; & n \geq 3 \end{cases}$$

$$\alpha(0) = \alpha(1) = I_n$$

$$t_n : I \times S^{n-1} \circlearrowleft$$

$$(t, \omega) \mapsto (t, \alpha(t)\omega)$$

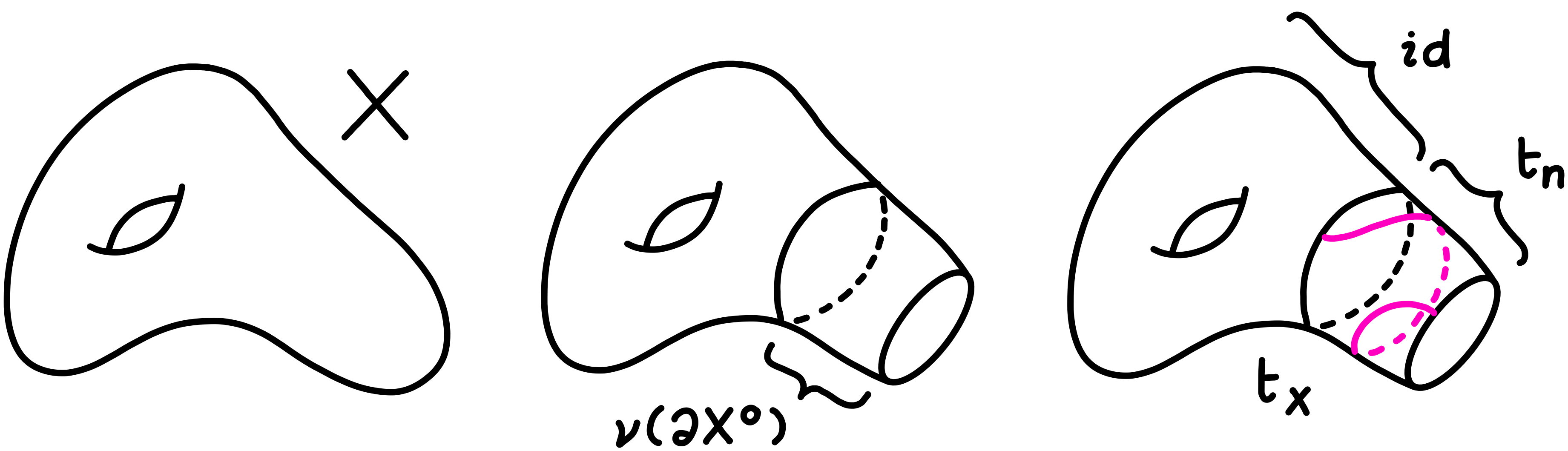


BOUNDARY DEHN TWISTS

$X^\circ := X \setminus B^n \Rightarrow$ Nbhd. $\nu(\partial X^\circ) \cong I \times S^{n-1}$

$t_X : X^\circ \circlearrowleft$

$t_X|_{\nu(\partial X^\circ) \cong I \times S^3} := t_n,\; t_X|_{X \setminus \nu(\partial X^\circ)} := id$



WHY STUDY ∂ DEHN TWISTS?

In dimension $n=4$:

Thm (Giansiracusa '08) t_X generates $\mathrm{Ker}(\pi_0 \mathrm{Diff}^+(X^\circ, \partial) \twoheadrightarrow \pi_0 \mathrm{Diff}^+(X))$

Thm (Orson-Powell '25) t_X is trivial in $\pi_0 \mathrm{Homeo}^+(X^\circ, \partial)$ $\Rightarrow$ when nontrivial in $\pi_0 \mathrm{Diff}^+(X^\circ, \partial)$, t_X is EXOTIC!

WHEN IS t_X EXOTIC?

t_X trivial in $\pi_0 \mathrm{Diff}^+(X^\circ, \partial)$ (NOT exotic): $X = \#_m (S^2 \times S^2)$, X not spin (Orson-Powell '25)

Nontrivial (exotic): $X \simeq_{htpy} K3$ (Baraglia-Konno '22, Kronheimer-Mrowka '20), $X = K3 \# (S^2 \times S^2)$ (J. Lin '23), $X = K3 \# K3$ (Tilton '25), $\forall X$ satisfying some condition (Ex: all complete intersections w/ $c_1 \equiv 0\,(32)$, $\sigma \equiv 16\,(32)$; many $E(4n-2)_{ij}$) (Baraglia-Konno '25)

WHERE DOES THIS EXOTICITY LIVE?

To get some insight, apply a natural operation. Does it kill the exoticity?

Ex: stabilizing K3 ($\#$ing w/ $S^2 \times S^2$) didn't (J. Lin '23)

How about abelianizing $\pi_0 \mathrm{Diff}^+(X^\circ, \partial)$?

$$\cdot^{ab} : \pi_0 \mathrm{Diff}^+(X^\circ, \partial) \longrightarrow (\pi_0 \mathrm{Diff}^+(X^\circ, \partial))^{ab}$$

Map killing all commutators
$[[f,g]] = [fgf^{-1}g^{-1}] \mapsto [id]^{ab} = 0$

$$\cong H_1(B\mathrm{Diff}^+(X^\circ, \partial))$$

Y. LIN: ABELIANIZATION KILLS t_{K3}

Q: When is $[t_X]^{ab}$ trivial?

Interesting: apps. to diffeos. & bundles

Thm (Y. Lin '25) $[t_{K3}]^{ab}$ is trivial

Pf: used deep results specific to K3:

Global Torelli + work of Baraglia-Konno ('22, '23)

Doesn't directly generalize...

OUR APPROACH $\Rightarrow [t_{K3}]^{ab} = 0$

(0) $K3 := \{[z] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_2 z_3^3 = 0\}$

(1) Make $a, c : K3 \circlearrowleft$ commuting diffeos. w/ certain behavior near $[0:0:0:1]$

(2) $K3^\circ := K3 \setminus \nu([0:0:0:1])$, twist

a, c near $\partial K3^\circ \rightsquigarrow a^\circ, c^\circ$ $= id$ very near $\partial K3^\circ$, $[a^\circ, c^\circ] = t_{K3}$

Constructs commutator equaling t_{K3}

$\Rightarrow [t_{K3}]^{ab}$ is trivial!

(1) BUILDING $a, c : K3 \circlearrowleft$ AS DESIRED

$$K3 := \{[z] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_2 z_3^3\}$$

$\Rightarrow A : [z] \mapsto [-z_0 : z_1 : z_2 : z_3]$, $C : [z] \mapsto [\bar{z}]$

rest. to $a, c : K3 \circlearrowleft$ commuting diffeos.

$\Rightarrow_{\text{Imp. func. thm.}}$ In coords. near $[0:0:0:1]$,

a, c act as $\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$

COMMUTATOR LOOPS IN SO(4)

FACT: For paths $R_a, R_c : \underbrace{I}_{:=[0,1]} \to SO(4)$,

from I_4 to $\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$,

$t \mapsto [R_a(t), R_c(t)]$ is a loop in $SO(4)$,

generates $\pi_1(SO(4)) \cong \mathbb{Z}/2\mathbb{Z}$

(2) FROM $a, c : K3 \circlearrowleft$ TO $a^\circ, c^\circ : K3^\circ \circlearrowleft$

$K3^\circ := K3 \setminus \nu([0:0:0:1])$ small enough nbhd of $[0:0:0:1]$

In coords. on $\nu(\partial K3^\circ)$ small enough nbhd of $\partial K3^\circ$,

$a = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = R_a(1)$, $c = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = R_c(1)$

Def $a^\circ, c^\circ : K3^\circ \circlearrowleft$ s.t. $a^\circ, c^\circ = id$ near $\partial K3^\circ$

Twist by R_a, R_c in $\nu(\partial K3^\circ)$

Set $a^\circ = a$, $c^\circ = c$ on $K3^\circ \setminus \nu(\partial K3^\circ)$

SO, WE HAVE $[a^\circ, c^\circ] = t_{K3}$

On $\nu(\partial K3^\circ) \cong I \times S^3$, $[a^\circ, c^\circ]$ takes

$$(t, \omega) \mapsto (t, [R_1(t), R_2(t)](\omega))$$

gens. $\pi_1(SO(4))$

On $K3^\circ \setminus \nu(\partial K3^\circ)$,

$$[a^\circ, c^\circ] = [a, c] = id$$

$\Rightarrow$ On all of $K3^\circ$, $[a^\circ, c^\circ] = t_{K3}$

$\cdot^{ab}$ kills commutators $\Rightarrow [t_{K3}]^{ab}$ is trivial

COMPLETE INTERSECTIONS

Homogeneous polys. $p_i : \mathbb{C}^{N+1} \to \mathbb{C}$, $i = 1, \ldots, M$

of degs. d_i (i.e., $p_i(\lambda z) = \lambda^{d_i} p_i(z)$)

$X_p := \{ [z] \in \mathbb{CP}^N \mid p_i(z) = 0 \ \forall i \}$ "$p_1, \ldots, p_M$ cut out X_p"

X_p a smooth cplete. int. if it's a

$\mathbb{C}$-mfld of $\mathbb{C}$-dim. $N-M$

OUR GENERAL CONSTRUCTION

(0) For such $p_1, \dots, p_M$, X_p $\left(\begin{smallmatrix}\dim_{\mathbb{C}} X = N-m \\ =: n/2 \text{ even}\end{smallmatrix}\right)$, $\exists$

"symmetric" $\begin{smallmatrix}\text{homog.}\\ \text{polys.}\end{smallmatrix}$ $q_1, \dots, q_M$ s.t. $X_q \cong_{\text{diffeo.}} X_p$

(1) Make $a, c : X \circlearrowleft$ commuting diffeos. w/ certain behavior near $[0 : \cdots : 0 : 1]$

using symmetries of $q_1, \dots, q_M$

(2) $X^\circ := X \setminus \nu([0 : \cdots : 0 : 1])$, twist

a, c near $\partial X^\circ \rightsquigarrow a^\circ, c^\circ$ $= id$ very near ∂X°, $[a^\circ, c^\circ] = t_X$

(0) PICKING SYMMETRIC CUTTING POLYS.

Thom: For $\substack{\text{homog.}\\ \text{polys.}}$ $q_1, \dots, q_M : \mathbb{C}^{N+1} \to \mathbb{C}$ s.t. $\deg q_i = \deg p_i$ for each $i = 1, \dots, M$, if X_q is a smooth cplete. int., $X_q \cong_{\text{diffeo.}} X_p$

Lemma (L.'26): For a generic choice of such $\substack{\text{homog.}\\ \text{polys.}}$ $q_1, \dots, q_M$ w/ $\begin{cases} \text{only even powers of } z_0 \\ \text{only } \mathbb{R} \text{ coeffs.} \\ \text{no term w/ only } z_N \end{cases}$, X_q is a smooth cplete. int. → pick any such $q_1, \dots, q_M$!

(1) BUILDING $a, c : X \circlearrowleft$ AS DESIRED

Polys. q_i have $\begin{cases} \text{only even powers of } z_0 \\ \text{only } \mathbb{R} \text{ coeffs.} \end{cases}$

$\Rightarrow A : [z] \mapsto [-z_0 : z_1 : \cdots : z_N]$, $C : [z] \mapsto [\bar{z}]$

rest. to $a, c : X_q \cong X_p \circlearrowleft$ comm. diffeos.

Polys. q_i have $\substack{\text{no term w/} \\ \text{only } z_N}$ $\Rightarrow [0 : \cdots : 0 : 1] \in X_q$

$\Rightarrow_{\text{Imp. func. thm.}}$ In coords. near $[0 : \cdots : 0 : 1]$,

a, c act as $\begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$, $\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{bmatrix}$

COMMUTATOR LOOPS IN SO(n)

FACT: For paths $R_a, R_c : \underbrace{I}_{:=[0,1]} \to SO(n)$,

from I_n to $\begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$, $\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \end{bmatrix}$

$t \mapsto [R_a(t), R_c(t)]$ is a loop in $SO(n)$,

generates $\pi_1(SO(\underset{\geq 3}{n})) \cong \mathbb{Z}/2\mathbb{Z}$

(2) FROM $a, c : X \circlearrowleft$ TO $a^\circ, c^\circ : X^\circ \circlearrowleft$

$X^\circ := X \setminus \nu([0 : \cdots : 0 : 1])$

Def $a^\circ, c^\circ : X_q^\circ \circlearrowleft$ s.t. $a^\circ, c^\circ = id$ near ∂X_q°

Twist by R_a, R_c in $\nu(\partial X_q^\circ)$

Set $a^\circ = a$, $c^\circ = c$ on $X_q^\circ \setminus \nu(\partial X^\circ)$

On all of $X_q \cong X_p$, $[a^\circ, c^\circ] = t_{X_q} \Rightarrow [t_{X_p}]^{ab}$ is trivial

Compatible w/ #s: For such $a_1, c_1 : X_1 \circlearrowleft$, $a_2, c_2 : X_2 \circlearrowleft$, can make $a^\circ, c^\circ : X^\circ := (X_1 \# X_2)^\circ$ s.t. $[a^\circ, c^\circ] = t_X$

SO, WE'VE PR□VED...

Thm (L.'26): For X any cplete int.

(w/ $\dim_{\mathbb{C}} X$ even) or # thereof, we can

build $a^\circ, c^\circ \in \mathrm{Diff}^+(X^\circ, \partial)$ s.t.

$$[[a^\circ, c^\circ]] = [t_x] \in \pi_0 \mathrm{Diff}^+(X^\circ, \partial)$$

$(\Rightarrow [t_x]^{ab} = 0)$

"Boundary Dehn twists are often commutators"

WHAT THIS SHOWS...

Known exotic ∂ Deh twists on K3 (see Y. Lin),

K3 # K3, K3 # ($S^2 \times S^2$), the ∞ many cplete ints. w/ $c_1 \equiv 0\ (32)$, $\sigma \equiv 16\ (32)$

are killed by abelianization!

We show the Thm for X $\in$ yet more general classes of spaces & their #s

$\supset$ the elliptic surfaces E(n) (for ex.)

WHAT NEXT?

- Commutators in Torelli $:= \mathrm{Ker}(\pi_0 \mathrm{Diff}^+(X,\partial) \to \mathrm{Aut}(H_2(X), Q_X))$

  ↳ ∂ Dehn twists shown by Baraglia-Konno ('25) to be nontrivial even have nontrivial image in the abelianization of Torelli!

- Commutators in $\mathrm{Symp}(X^\circ, \omega, \partial)$
- Other Dehn twists along Seifert fibered spaces $\partial X \neq S^3$
- New nontriviality results