
Aya
TALK

KEY:

Only spoken

Only written

Spoken + written

Instruction

Draw this, describe while drawing

Draw this, don't describe while drawing

LENGTH: 1HR

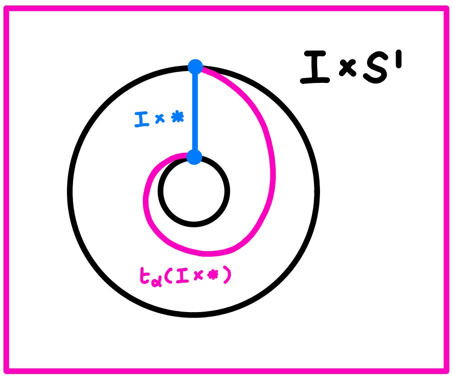
HELLO! I'm Ayo and today I'll be telling you about certain diffeomorphisms called Boundary dehn twists which have seen a major wave of interest in the past half decade, especially in dimension 4. The headline result I'll be presenting about is that these boundary Dehn twists are often commutators in the smooth mapping class group rel ∂ $\pi_0(\mathrm{Diff}^+(X^\circ,\partial))$ of punctured X, where X is a SM ori. closed n-mfld, and $X^\circ := X \setminus B^n$. To unpack what this is saying, let's formally introduce this mapping class group and the boundary Dehn twist. Specifically, we denote by $\mathrm{Diff}^+(X) :=$ the group of $\{$ori.-pres. diffeos. $X \circlearrowleft\}$ and by $\mathrm{Diff}^+(X^\circ,\partial) :=$ the group of $\{$ori.-pres. diffeos. $f: X^\circ \circlearrowleft \mid f|_{\partial X^\circ} = id\}$, which we call the diffeomorphism group rel ∂ of X°. These groups are super natural to study, but they're a little big! Their groups of path components $\pi_0\mathrm{Diff}^+(X)$ and $\pi_0\mathrm{Diff}^+(X^\circ,\partial)$ are a whole lot smaller, which is good if we want to compute anything about them, and they are also just generally super important, so they get special names! Specifically, they're called the $=:$ smooth mapping class group of X and the $=:$ smooth MCG rel ∂ of X°. Elements of these groups are then called smooth mapping classes and smooth mapping classes rel ∂, and are often called smooth isotopy classes and smooth isotopy classes rel ∂ of diffeomorphisms of X and X°.

One can also consider the topological mapping class group of X and topological mapping class group rel ∂ of X°; the analogous groups where we replace the word diffeomorphism point with the word homeomorphism, but we'll mostly focus on the smooth setting today.

Now, let's think of how we can make a nontrivial element of some smooth mapping class group rel ∂. There's a super natural way to do this in the simple case when $X = D^n$, so $X^\circ \cong (I := [0,1]) \times S^{n-1}$. Specifically, for a smooth loop $\alpha : I \to SO(n)$ w/ $\alpha(0) = \alpha(1) = I_n$, we get a $\rightsquigarrow$ diffeomorphism rel boundary $t_\alpha \in \mathrm{Diff}^+(I \times S^{n-1}, \partial)$ of $I \times S^{n-1}$ by setting $t_\alpha(s,\omega) := (s, \alpha(s)\omega)$, so we apply a different rotation $\alpha(s)$ in each S^{n-1} cross-section of $I \times S^{n-1}$, and this rotation is the identity on each ∂ component. A path in the space of loops in $SO(n)$ based at the identity from α to the constant identity loop then, by this construction, gives a path in the smooth mapping class group rel ∂ of X° from t_α to the identity diffeomorphism, so $[\alpha] = 1 \in \pi_1(SO(n)) \Rightarrow [t_\alpha] = 1 \in \pi_0 \mathrm{Diff}(I \times S^{n-1}, \partial)$.

In fact, we can visualize why the converse is also true $\boxed{\Leftrightarrow}$, so any loop α in $SO(n)$ based at the identity which is

homotopically nontrivial will give rise to a diffeomorphism t_α of the punctured n-disk whose smooth mapping class rel ∂ is nontrivial. Let's see this when $n=2$., We then have that the loop $\alpha : t \mapsto e^{2\pi i t}$ gens. $\pi_1(SO(2)) \cong \mathbb{Z}$.
We can then just draw the resulting diffeomorphism t_α of $I \times S^1$:



Now, any path in the diffeomorphism group rel boundary from t_α to the identity would certainly give a smooth deformation from this arc point to the other one point fixing the endpoints of the arcs, but just looking at these arcs, we can see that no matter how hard we wiggle them keeping their endpoints fixed, we can't wiggle them into one another. Therefore, no such path can exist, so the smooth mapping class rel ∂ of t_α can't be trivial. This intuition directly extends to all higher dimensions, where we note that $\pi_1(SO(n \geq 3)) \cong \mathbb{Z}/2$ rather than $\mathbb{Z}$. Then, For $[\alpha]$ a gen. of $\pi_1(SO(n))$, we call $[t_\alpha] \neq 1 \in \pi_0 \mathrm{Diff}^+(I \times S^{n-1}, \partial)$ the Dehn twist t_n on $I \times S^{n-1}$.

Now, from this very natural diffeomorphism, we can make many more, which then may or may not be nontrivial in the smooth mapping class group rel ∂. To do this, let's consider a SM closed ori n-mfld X, and its puncture $X^\circ := X \setminus B^n$.

Because X is closed, $\partial X^\circ \cong S^{n-1} \Rightarrow \nu(\partial X^\circ) \cong I \times S^{n-1}$.

We can then Consider $t_X \in \mathrm{Diff}^+(X^\circ, \partial)$ given by applying the Dehn twist t_n in this point neighborhood of the ∂

$t_X|_{\nu(\partial X^\circ) \cong I \times S^{n-1}} := t_n$, and the identity otherwise

$t_X|_{X^\circ \setminus \nu(\partial X^\circ)} = id$. t_X is then called the $=:$ Boundary Dehn twist on X°. These are very special diffeomorphisms - for example, a

Thm (of Giansiracusa from '08) states that

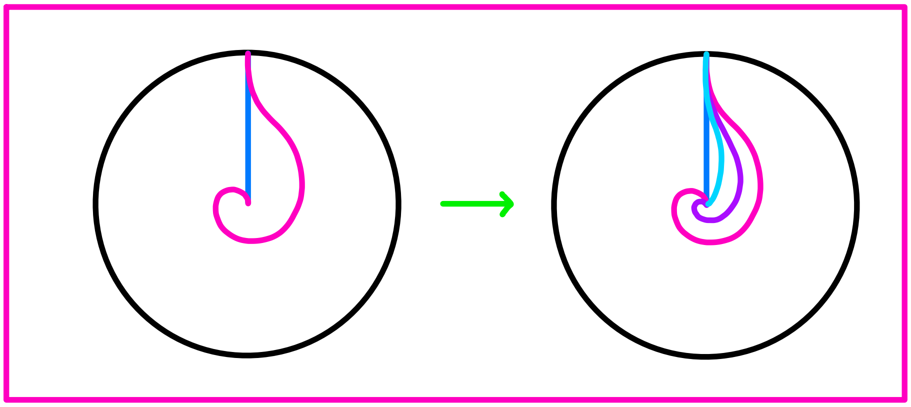
For $n=4$ (which is our main setting of interest),

t_X gens. $\mathrm{Ker}(\pi_0 \mathrm{Diff}(X^\circ, \partial)) \twoheadrightarrow \pi_0 \mathrm{Diff}(X)$

given by extending a diffeomorphism on punctured X to the identity on the 4-ball we removed from X to get the puncture, so this kernel is trivial if the boundary Dehn twist is trivial and $\mathbb{Z}/2$ otherwise.

The importance of these diffeomorphisms has motivated great interest in the Question: when is t_X trivial in the smooth or topological mapping class group rel ∂?

Let's first visualize a simple case. Taking $X = S^2$, we can easily draw a picture:



Now, this argument easily extends to all dimensions, so we can see that $\Rightarrow$ t_{S^n} is trivial in the smooth and topological categories. In fact, in our favorite setting of dimension 4, this is the norm in the topological category:

a Thm (of Orson, and Powell from '25)

says that $\forall$ closed X^4 w/ $\pi_1 X \cong 1$, t_X is trivial in the topological mapping class group rel ∂ of X° $\pi_0(\mathrm{Homeo}(X^\circ, \partial))$.

This behavior is also seen in the smooth category when X is not spin: formally, another

Thm (of Orson, and Powell from '25) says that

If such X is not spin, t_X is trivial in $\pi_0 \mathrm{Diff}(X^\circ, \partial)$.

However, Much less is known when X is spin:

I'll survey the short list of what are, as far as I know, all known results in this setting: In terms of results known when t_X is trivial in $\pi_0 \mathrm{Diff}(X^\circ, \partial)$: the only cases I know are when X is some $\#(S^2 \times S^2)$s.

And as for cases where it's known that it's Nontrivial:

- This is known for $X \cong_{\text{Homeo}} K3$ surface (by Baraglia, and Konno '22, and Kronheimer, and Mrowka in '20),
- for $X =$ the once-stabilized K3 surface $K3 \# (S^2 \times S^2)$ (by J.ianfeng Lin in '23),
- for $X = K3 \# K3$ (by Tilton very recently '25)
- and $\forall X$ satisfying a technical condition (by Baraglia, and Konno in '25). Principal examples of manifolds satisfying this condition are
  - ↳ Many double log transforms of the elliptic surfaces $E(4n-2)_{i,j}$ and
  - ↳ all Complete intersections w/ $c_1 \equiv 0$, $\sigma \equiv 16 \pmod{32}$.

We'll talk more about these later.

Being topologically isotopic but not smoothly isotopic to the identity means by definition that These are what we call exotic diffeos. You might've heard this buzz-word "exotic" before because the study of exotica — that being, structures which are topologically equivalent but smoothly distinct - is one of the central pursuits in 4-manifold topology, where everything is super messed up and exotic behavior runs rampant. So, the study of boundary dehn twists is very much motivated by the goal of studying natural instances of this wild and somewhat counterintuitive mathematical phenomenon. Something that I personally find super interesting is trying to track down exactly what fundamentally causes this exotic behavior. A way to gather some information about this is to consider whether a given instance of exotic behavior stays exotic or becomes trivial after the application of some natural operation. A common such operation is to connected sum the manifold on which this exoticity takes place with $S^2 \times S^2$; we call this "stabilization" of the manifold. Note we already discussed how applying this stabilization operation to the K3 surface fails to kill the exotic ∂ Dehn twist point on its puncture; this fact then says something interesting about what causes the wacky exotic behavior of this ∂ Dehn twist (and specifically, that it must be caused by something deep enough that stabilization, which frequently kills exotic behavior, can't in this case). Now, another interesting natural operation one could consider is what we call Abelianization: specifically, we can consider the "abelianization" map

$$\pi_0 \mathrm{Diff}^+(X^\circ, \partial) \xrightarrow{ab} (\pi_0 \mathrm{Diff}^+(X^\circ, \partial))^{ab}$$

which takes all commutators in the smooth mapping class group to the identity $[f,g] \mapsto 1$, therefore making the group abelian.

It's then interesting, in determining what causes the exoticity of these ∂ Dehn twists, whether their images under this abelianization map - called abelianized ∂ Dehn twists - are trivial. This was formalized as a

Question of (Y. ujie Lin in '25), which specifically asked:

For which simply-connected closed X^4 is the abelianized boundary Dehn twist $t_X^{ab} \in (\pi_0 \mathrm{Diff}^+(X^\circ, \partial))^{ab}$ trivial?

Now, outside of what an answer to this question would say about the exotic behavior of the ∂ Dehn twist, it's also of interest in its own right: specifically, this is due to the relationship between the abelianized smooth mapping class group and

$$\boxed{(\pi_0 \mathrm{Diff}^+(X^\circ, \partial))^{ab} \longleftrightarrow \text{deg-1 char. classes of smooth } X\text{-bundles}}$$

via an isomorphism between this abelianization and $\cong H_1(B\mathrm{Diff}^+(X^\circ, \partial))$. We don't have time to get into the specifics of the beautiful picture of this association, but just note that abelianized mapping class groups have been of great interest to mathematicians for decades, in part because information about these groups - for example, whether or not incredibly natural elements they contain such as abelianized ∂ Dehn twists are trivial – will give us interesting information about smooth bundles.

That's one reason I was excited to see Lin's

Thm (Y. Lin '25) that t^{ab}_{K3} is trivial.

Now, Lin's Pf used some really deep results about the K3 surface: in the form of

- the Global Torelli Thm and
- An obstruction of Baraglia - Konno ('22) for spin families.

It's worth noting that aspects of

these deep results are quite specific to the K3 surface, so there really isn't much room for them to be directly generalized to other settings of interest.

Now, that's where the mathematics we'll discuss in

The rest of this talk comes in:

Specifically we'll give an Overview of our broadly applicable concrete realization of t_X as a commutator $[a^\circ, c^\circ]$ for $a^\circ, c^\circ \in \pi_0 \mathrm{Diff}^+(X, \partial)$, which we note (gives that $\Rightarrow t_X^{ab} = 0$)

We'll break this overview into 3 parts:

We'll first (I) Give our construction for $X = K3$, which serves as a great setting to get acquainted with our construction.

We'll then (II) Sketch our construction for an important class of mflds. X, incl. many of the spaces point on which we know $t_X \neq 0$

(so, this hints that when the ∂ Dehn twist is nontrivial, its exoticity is often killed by abelianization, saying something about what could cause that exotic behavior), and we'll finally (III) State our general Thm

Now, to begin with this (I)st point, one may ask the question: What is the K3 surface? Well, there's a lot of great ways to answer that question, but to start, let's note it's The unique SM closed $\pi_1 = 1$ 4-mfld satisfying a certain natural condition. Note that there's tons of non-isomorphic complex and algebraic manifolds satisfying this condition which are all called K3 surfaces, but a result of Kodaira says they're all diffeomorphic, so to us differential topologists, there's only one. Now, this definition here point is cool but super far from concrete. A really concrete way to realize the K3 surface is as $K3 \cong$ the zero locus in $\mathbb{CP}^3$ of a generic degree 4 homogeneous poly. on $\mathbb{C}^4$. To say this in more detail, we consider $\mathbb{CP}^3$ as $:=$ the set of $\mathbb{C}$ lines in $\mathbb{C}^4$ $:= (\mathbb{C}^4 \setminus 0) / x \sim \lambda x \ \forall \lambda \in \mathbb{C}$

A Homog. poly. $p: \mathbb{C}^4 \to \mathbb{C}$ of deg. d is then a polynomial which satisfies $p(\lambda x) = \lambda^d p(x) \ \forall x \in \mathbb{C}^4, \lambda \in \mathbb{C}$

Now, from this homogeneity condition, we can see that $\Rightarrow p(x) = 0$ iff $p(x') = 0 \ \forall x' \in [x] \in \mathbb{CP}^3$, so

$\Rightarrow$ The zero locus $X_p := \{[x] \in \mathbb{CP}^n \mid p(x) = 0\}$ is well-defined. For a generic choice of polynomial p, X_p will then, in fact, be a SM submanifold of $\mathbb{CP}^3$. This construction and mild generalizations thereof have turned out to be fantastic sources of important smooth manifolds, an important example of which is the K3 surface, which we noted arises taking p to have degree 4. To work with a specific representative of the K3 surface, we fix $p := x_0^4 + x_1^4 + x_2^4 + x_2 x_3^3$ and set, $K3 := X_p$. One can then check that this is the smooth simply-connected 4-manifold we want it to be. Note that we're using a somewhat funny degree 4 homogeneous polynomial here to define K3; in fact, this polynomial has certain symmetries that we'll now make use of to define certain involutions of interest on the K3 surface, which we will then use to get a concrete commutator representative for the ∂ Dehn twist on the punctured K3 surface. Specifically, consider the maps $A, C : \mathbb{C}^4 \circlearrowleft$, defined by taking some z in $\mathbb{C}^4$ to $A(z) := (-z_0, z_1, z_2, z_3)$, and to $C(z) := \bar{z}$. Now, first note that because p has only even powers of z_0, we have that $\Rightarrow$ $p(Az) = z$, so $\Rightarrow$ $p(Az) = 0$ iff $p(z) = 0$. Therefore, by our definition of the K3 surface as, $\Rightarrow$ the map $A' : [z] \mapsto [Az]$ on $\mathbb{CP}^3$ restricts to a map $a := A'|_{K3} : K3 \circlearrowleft$. Moreover, because p has only $\mathbb{R}$-coeffs., we have that $\Rightarrow p(C(z)) = \overline{p(z)}$, so $\Rightarrow p(Cz) = 0$ iff $p(z) = 0$. Similarly, we then have that $\Rightarrow$ the map $C' : [z] \mapsto [Cz]$ on $\mathbb{CP}^3$ restricts to a map $c := B'|_{K3} : K3 \circlearrowleft$.

Now, a, c commute because A and C commute, and they also have the shared fixed point $x := [0:0:0:1]$, which we can observe lies $\in K3$

b/c p has no term w/ only the last coordinate function x_3.

It's then $\Rightarrow$ an exercise in the use of the implicit func. thm. to show that

$\exists$ loc. coords $\varphi : U \subset K3 \to \mathbb{C}^2$, $\varphi(x) = 0$ s.t.

$\varphi^{-1} \circ a \circ \varphi(w) = (-w_0, w_1)$, $\varphi^{-1} \circ c \circ \varphi(w) = \overline{w}$.

Now, we're well on our way to providing a concrete commutator representative for the ∂ Dehn twist on the punctured K3

surface in the smooth MCG rel ∂! To do this, let's make a certain

Observation: For $R_a, R_c : [1,2] \to SO(4)$ paths satisfying

$$R_a(1) = R_c(1) = 1, \quad R_a(2) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} (= \varphi^{-1} \circ a \circ \varphi),$$

$$R_c(2) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} (= \varphi^{-1} \circ c \circ \varphi),$$

the loop $R: s \mapsto [R_a(s), R_c(s)]$ satisfies $R(0) = R(1) = I_4$ and $[R]$ gens. $\pi_1(SO(4))$. This is just a little exercise using the spin group, the universal cover of the special-orthogonal group. Unfortunately, we don't have time to do this exercise today, so I hope you can believe me that it's super straightforward. Now, if you can find it in your heart to trust me on that, I'm happy to confirm that

we're almost there! We then set $K3^\circ := K3 \setminus \varphi^{-1}(B_1)$, so $a|_{K3^\circ}, c|_{K3^\circ} : K3^\circ \circlearrowleft$, not fixing $\partial K3^\circ$. Then, we Consider $a^\circ, c^\circ : K3^\circ \circlearrowleft$ which coincide with a and c respectively outside of $\varphi^{-1}(B_2)$ and twist by R_a and R_c in $\varphi^{-1}(B_2 \setminus B_1)$, so we specifically have

$$a^\circ(y) := a(y),\ c^\circ(y) := c(y)\ \forall y \in K3^\circ \setminus \varphi^{-1}(B_2 \setminus B_1),$$

$$a^\circ(y) := \varphi^{-1} \circ R_a(|\varphi(y)|) \circ \varphi(y),\ c^\circ(y) := \varphi^{-1} \circ R_a(|\varphi(y)|) \circ \varphi(y) \quad \forall y \in \varphi^{-1}(B_2 \setminus B_1),$$

Are there any questions about what's going on here?

Pause

Great! So, we can then see that $a^\circ, c^\circ \in \mathrm{Diff}^+(K3^\circ, \partial)$, since our smooth paths $R_a(s)$ and $R_c(s)$ coincide with a and c in coordinates when $s = 2$, allowing us to guarantee smoothness, and equal the identity when $s = 1$, which here point shows that a° and c° restrict to the identity on the ∂.

Now, let's consider the Commutator

$[a^\circ, c^\circ] = a^\circ \circ c^\circ \circ (a^\circ)^{-1} \circ (c^\circ)^{-1}$ of these

diffeomorphisms. Well, a, c commute, so by our definition

of a° and c°, $\Rightarrow [a^\circ, c^\circ]|_{K3^\circ \setminus \varphi^{-1}(B_2 \setminus B_3)} = id$. Moreover,

we can also see from this definition that

On $B_2 \setminus B_1$, $\varphi \circ [a^\circ, c^\circ] \circ \varphi^{-1}(v) = [R_a, R_c](|v|)v = R(|v|)v$.

Then, Identifying $B_2 \setminus B_1 \cong I \times S^3$, we have that

$\varphi \circ [a^\circ, c^\circ] \circ \varphi^{-1}(s, \omega) = (s, R(s)\omega)$, where we note our

Observation said that $\Rightarrow [R]$ gens. $\pi_1(SO(4))$.

Any questions about any of this?

Pause

Great! Well, now that we have this description of how our commutator acts, let's think back to our definition of the ∂ Dehn twist on $K3^\circ$. We defined it to act as rotation by a loop that generates $\pi_1(SO(4))$ in a neighborhood of the boundary, and by the identity elsewhere. Well gosh, that's exactly what this commutator does! So, we get that

$[[a^\circ, c^\circ]] = [t_{K3}] \in \pi_0 \mathrm{Diff}^+(K3^\circ, \partial)$, so notably, $\Rightarrow t_{K3}^{ab} = 0$

since we mentioned that the abelianization map kills all commutators.

Pause for questions

Now, let's briefly sketch our general argument that formalizes an analogous commutator presentation for ∂ Dehn twists on the punctures of an important class of manifolds:

(II) Complete intersections. Recall how we realized K3 as the zero set of a homogeneous degree 4 polynomial satisfying certain symmetries. Generalizing this, we

Define a SM mfld $X \subset \mathbb{CP}^n$ to be is a smooth complete intersection if X is the zero locus of m homog. polys. $p_1, \dots, p_m : \mathbb{C}^{n+1} \to \mathbb{C}$ and $\dim_{\mathbb{C}} X = n-m$, so we previously realized K3 as a complete intersection for $n=3$ and $m=1$.

Any questions about this?

Pause

Now, let's think through how to translate what we did for the K3 surface to this setting! Well, first, we exploited certain Symmetries we'll now call (✱)

of the polynomial p we used to define K3: to descend involutions A' and C' from $\mathbb{CP}^3$ to K3. Specifically, these symmetries were that

- p has only even powers of z_0 (which we used to show that $\Rightarrow$ the map $A' : [z] \mapsto [-z_0, z_1, \dots, z_{n=3}]$ on $\mathbb{CP}^{n=3}$ rest. to $a : K3 \circlearrowleft$), that
- p has only $\mathbb{R}$ coeffs. (which we used to show that $\Rightarrow$

the map $B' : [z] \mapsto [\bar{z}]$ on $\mathbb{CP}^{n=3}$ rest to $b : K3 \circlearrowleft$), and that

· p has no term w/ only the last coordinate function $x_{n=3}$ (which we used to show that $\Rightarrow x := [0 : \cdots : 0 : 1] \in X$, so a, c have x as a shared fixed pt.)

Then, we punctured $X = K3$ at x to get $X^\circ = K3^\circ$, and twisted $a|_{X^\circ}, c|_{X^\circ}$ to get $a^\circ, c^\circ \in \mathrm{Diff}^+(X^\circ = K3^\circ, \partial)$ s.t. $[[a^\circ, c^\circ]] = [t_{X=K3}] \in \pi_0 \mathrm{Diff}^+(X^\circ = K3^\circ, \partial)$.

Now, I've been suggestively writing X here to hint towards the fact that we actually formalized this process for any complete intersection X! The way we do this builds off of an Observation (of Thom from around the middle of the last century) that Picking a SM cplete int. $X \subset \mathbb{CP}^n$ which is the, zero locus of homog. polys. $p_1, \ldots, p_m$, a generic choice of homog. polys. $q_1, \ldots, q_m$ w/ $\deg q_i = \deg p_i$ will have zero locus $X' \cong_{\text{Diffeo.}} X$. Now, this is super cool because it says that the smooth topology of a complete intersection X only depends on the degrees of the polynomials we use to define X. However, we can't yet directly apply the approach we did for the K3 surface because of this pesky word generic-a priori, because the set of polynomials exhibiting these symmetries (*) point is very small in the space of all polynomials, it could a priori be the case that all polynomials exhibiting these symmetries have singular zero loci, despite this genericity point Thom so generously gives us. But, in fact,

We show that we don't need to worry about this: For such X, a generic choice of homog. polys. $q_1, \dots, q_m$ w/ $\deg q_i = \deg p_i$ each satisfying the symmetries $(\ast)$ will have zero locus $X' \cong_{\text{Diffeo.}} X$. Then, thinking of the smooth manifold X as the zero locus of these homogeneous polynomials q_i,

$\Rightarrow$ the $_{\text{Above argument}}$ gives us $a, c : X \circlearrowleft$ w/ shared fixed pt. x as above, which then $\Rightarrow$, again, using the exact same puncturing and twisting argument used $_{\text{Above argument}}$, we can construct $a^\circ, c^\circ \in \mathrm{Diff}^+(X^\circ, \partial)$ s.t. $[[a^\circ, c^\circ]] = [t_x] \in \mathrm{Diff}^+(X^\circ, \partial)$ (so, $\Rightarrow t_x^{ab} = 0$).

There's the slight caveat that this c° will only preserve the orientation on X when $\dim_{\mathbb{C}} X$ is even, though of course, this isn't an issue in our favorite setting when X has real dimension 4.

Pause for questions

In fact, our concrete argument plays well with connected sums, so we've Also showed this for X any # of cplete. ints.

Alright, now onto the final (brief) part of this talk:

(III) What have we proved? Well, we've verified the Thm (L. '26) that For X any cplete int. of even $\mathbb{C}$ dimension or any # thereof, we may const. $a^\circ, c^\circ \in \mathrm{Diff}^+(X^\circ, \partial)$ s.t. $[[a^\circ, c^\circ]] = [t_x]$ $\in \mathrm{Diff}^+(X^\circ, \partial)$ (so, $t_x^{ab} = 0$)

This then shows that $\Rightarrow$ the Boundary Dehn twists known to be nontrivial in $\pi_0 \mathrm{Diff}^+(X^\circ, \partial)$ often become trivial after abelianization.

We also Prove the Thm for yet broader classes of spaces X which then Include the Elliptic surfaces $E(n)$, some of which are among the most natural candidates for future study of ∂ Dehn twists.

In proving all this, we also show facts about smooth bundles: specifically, $\forall X$ for which we show the Thm, with a and c the smooth involutions on X we built when proving the theorem, the mapping torus of a, c is a SM ori X-bundle $\to T^2$ w/non-spin total space. The existence of such bundles was really not clear to me before starting this project, so it's nice to know they're out there and how we can build them! So yeah, this is a fun little bonus result on top of what our work says about boundary Dehn twists! With that, I wanted to thank you all for listening, and to invite anyone who's interested in this stuff to chat with me after the talk or, if you miss me then, to email me later! I'd love to discuss any of what I've had the pleasure of telling you about, the further directions I'm now investigating, or anything else topology or otherwise.

And for anyone who'd just like to read more, I'm happy to say the paper formalizing what I've said today is, as of very recently, fully done and should be up on arXiv by this Wednesday evening.