ASYMPTOTICALLY SHORT GENERALIZATIONS OF t-DESIGN CURVES

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Abstract. Ehler and Gröchenig posed the question of finding t-design curves—curves whose associated line integrals exactly average all degree at most t polynomials—γ_t on S^d of asymptotically optimal arc length $\ell(\gamma_t) \asymp t^{d-1}$ as $t \to \infty$. This work investigates analogues of this question for *weighted* and *ε_t-approximate t-design curves*, proving existence of such curves γ_t on S^d of arc length $\ell(\gamma_t) \asymp t^{d-1}$ as $t \to \infty$ for all $d \in \mathbb{N}_+$ in the weighted setting (in which case such curves are asymptotically optimal) and all odd $d \in \mathbb{N}_+$ in the approximate setting (where we have $\varepsilon_t \asymp 1/t$ as $t \to \infty$). Formulas for such weighted t-design curves for $d \in \{2, 3\}$ are presented.

1. Background and main results

Spherical t-designs were introduced by Delsarte, Goethals, and Seidel [9] to be finite subsets of spheres such that any polynomial of degree at most t has the same average on a t-design as on the entire sphere. These objects are of interest for providing a notion of a "good" finite approximation of the sphere and for their connections to other areas of combinatorics [3], data analysis on the sphere, numerical analysis, and beyond. Of special interest are optimally small t-designs, which generally prove difficult to construct. Existence of t-designs on S^d for all t and d was proven by Seymour and Zaslavsky [27, Corollary 1], but the size of the smallest t-design on S^d is open in almost all cases. Delsarte, Goethals, and Seidel provided lower bounds of asymptotic order t^d as $t \to \infty$ on the sizes of spherical t-designs on S^d for all $t, d \in \mathbb{N}$ [9, Definition 5.13], but even the asymptotic order of size of an optimally small sequence $(X_t)_{t=1}^{\infty}$ of t-designs on S^d as $t \to \infty$ for $d > 1$ was a major open problem in the field for decades. Progress during this period included work of Mhaskar, Narcowich, and Ward [23] which showed that there exist *weighted* t-designs $(W_t)_{t=0}^{\infty}$ on S^d (with almost equal weights) satisfying $|W_t| \asymp t^d$ as $t \to \infty$. The problem was finally resolved in the unweighted setting when Bondarenko, Radchenko, and Viazovska showed that the lower bounds [9, Definition 5.13] of Delsarte, Goethals, and Seidel are asymptotically optimal up to constants depending only on d, proving that there exist constants $C_d > 0$ depending only on d such that for any

$C > C_d$ and $t \in \mathbb{N}_+$, there exists a (unweighted) t-design on S^d of size Ct^d [4, Theorem 1].

Motivated by the applications curves have had to experimental design and data analysis on the sphere through techniques such as mobile sampling [12, Section 1], Ehler and Gröchenig introduced *spherical t-design curves*, curves such that any polynomial of degree at most t has the same average value along the curve as on the entire sphere [12, Definition 2.1]. In analogy with spherical t-designs (henceforth called spherical t-design *sets*), particularly short t-design curves are of special interest in this setting. Ehler and Gröchenig showed that for any sequence $(\gamma_t)_{t=1}^{\infty}$ of t-design curves on S^d, the length $\ell(\gamma_t)$ of γ_t satisfies $\ell(\gamma_t) \gtrsim t^{d-1}$ (i.e. there exists a constant c_d depending only on d such that $\ell(\gamma_t) \geq c_d t^{d-1}$). These authors called sequences achieving asymptotic equality up to a constant depending only on d in this bound *asymptotically optimal* and posed the problem of proving existence of asymptotically optimal sequences of t-design curves on S^d, which they solved for $d = 2$ using the $d = 2$ case of the aforementioned result of Bondarenko, Radchenko, and Viazovska [4, Theorem 1]. The present author solved this problem for $d = 3$ [22, Theorem 1.2], but it remains open in all other dimensions, motivating the exploration of analogues of this problem concerning classes of curves which could find use in the potential data analysis and experimental design applications spherical t-design curves were introduced to have. Recently, Ehler, Gröchenig, and Karner presented results concerning such a problem, proving existence of sequences $(\gamma_t)_{t=1}^{\infty}$ of curves (specifically, geodesic cycles) satisfying Marcinkiewicz-Zygmund inequalities [13, Property (i) in Theorem 1.1] of asymptotic order of length at most t^{d-1} as $t \to \infty$ [13, Theorem 1.1], matching the known lower bound [12, Theorem 1.1] on the asymptotic order of length of a sequence of spherical t-design curves as $t \to \infty$. The main results of this manuscript, communicated by Theorems 1.3 and 1.6, are that there exist sequences indexed in t of generalizations of t-design curves we call *weighted* and *ε_t-approximate t-design curves* achieving this asymptotic order of arc length on all spheres in the weighted setting (in which case this asymptotic order of arc length is optimal) and all odd-dimensional spheres in the approximate setting (in which case we have $\varepsilon_t \asymp 1/t$ as $t \to \infty$).

Consider $d \in \mathbb{N}_+ := \{1, 2, ...\}$, $t \in \mathbb{N} := \{0, 1, ...\}$, and a continuous, piecewise smooth, closed curve $\gamma : [0, 1] \to S^d$ with finitely many self-intersections. Take $P_t(S^d)$ to be the space of restrictions to S^d of real-valued polynomials on $\mathbb{R}^{d+1}$ of degree at most t, where such a polynomial is an element of the span over $\mathbb{R}$ of products of at most t coordinate functions $\mathbb{R}^{d+1} \to \mathbb{R}$. Also take σ to be the uniform measure (meaning on a measure on a metric space with respect to which any two balls of the same radius have the same measure) on S^d, normalized so that $\sigma(S^d) = 1$.

Definition 1.1. We say that γ is a *weighted t-design curve* if

$$\int_0^1 f(\gamma(s))\,ds = \int_{S^d} f\,d\sigma$$

for all $f \in P_t(S^d)$.

Definition 1.2. For any $\varepsilon \geq 0$ and $c > 0$, we say that γ is an (ε, c)-*approximate t-design curve* if

$$\left| c\int_\gamma f - \int_{S^d} f\,d\sigma \right| \leq \varepsilon \left| \sup_{S^d} f \right|$$

for all $f \in P_t(S^d)$, where

$$\int_\gamma f := \int_0^1 f(\gamma(s))|\gamma'(s)|\,ds.$$

If $c = 1/\ell(\gamma)$ (where $\ell(\gamma) := \int_\gamma 1$ is the length of γ), we say that γ is an *ε-approximate t-design curve.*

In analogy with work of Ehler, Gröchenig, and Karner [13], we call a weighted or (ε, c)-approximate t-design curve which is also a *geodesic cycle*—meaning that its image is a union of finitely many geodesics on S^d—a *weighted* or *(ε, c)-approximate t-design cycle.*

Weighted spherical t-design sets—and, more generally, *quadrature* and *cubature* formulas—have received substantial study [15, 31, 30, 28, 25, 29, 7, 8]. Such objects are often defined as a pair of a finite subset of a sphere and positive weight function on this set, but in the curves setting, we encode the information of a "weighting" in the parametrization of the curve. Therefore, t-design curves are exactly curves which, when reparametrized to have constant speed, are weighted t-design curves.

To the knowledge of the author, the notion of approximate t-design sets analogous to that of Definition 1.2 has seen less study than that of weighted t-design sets: in fact, other inequivalent notions of "approximate" spherical t-design sets which have been investigated [33] are specifically weighted t-design sets with "approximately equal" weights. The present author is not aware of investigation of the notion of an approximate t-design set or curve described in this work prior to previous work [22, Section 3] of the author in which Theorem 1.6 was stated without proof as an example of further work to be done past the main result of that paper [22, Theorem 1]. ε-approximate t-design sets were recently independently introduced and first formally explored alongside another novel notion of approximate t-design sets in work of Dillon [10, Subsection 6.1].

For any $d \in \mathbb{N}_+$ and $\varepsilon > 0$, it is straightforward to construct the natural notion of a *weighted ε-approximate t-design curve* on S^d with no singularities by considering a curve $[0, 1] \to S^d$ such that $\gamma(s)$ is in a small enough neighborhood of a t-design set for all s in a subset of $[0,1]$ having measure close enough to 1, so we only consider these two generalizations of t-design curves separately. We first discuss the main results of this work concerning weighted t-design curves, then those concerning approximate t-design curves.

Theorem 1.3 (Main theorem on weighted t-design curves)**.** *For any $d > 1$, there exists a constant $C_{d-1} \in 2\mathbb{Z}$ such that for any $t \in \mathbb{N}_+$ and $C \geq \pi C_{d-1} t^{d-1}$, there exists a weighted t-design curve on S^d of length C. Fixing $t \in \mathbb{N}$ and considering any continuous, piecewise smooth map $\theta_1 : [-1,1] \to \mathbb{R}$, such a curve is given by the weighted $(2t-1)$-design curve*

$$\begin{aligned}
&(\alpha_1, \alpha_2, \alpha_3) : [0,1] \to S^2, \\
&\alpha_1(s) = (-1)^{\lfloor 2ts \rfloor}(4ts - 2\lfloor 2ts \rfloor - 1), \\
&\alpha_2(s) = \sqrt{1-\alpha_1(s)^2}\cos(\pi\lfloor 2ts \rfloor/t + \theta_1(\alpha_1(s))), \\
&\alpha_3(s) = \sqrt{1-\alpha_1(s)^2}\sin(\pi\lfloor 2ts \rfloor/t + \theta_1(\alpha_1(s)))
\end{aligned} \tag{1}$$

for $d = 2$ and $C \geq 2\pi t$ (with equality if θ_1 is constant). When $\theta_1(0) \neq m\pi - 2n\pi/t$ for all $m, n \in \mathbb{Z}$ and considering any continuous, piecewise smooth map $\theta_2 : [0,1] \to \mathbb{R}$ such that $\theta_2(0) - \theta_2(1) \in 2\pi\mathbb{Z}$, such a curve is given by the weighted $(4t-1)$-design curve

$$\begin{aligned}
&[0,1] \to S^3 \subset \mathbb{C}^2 \\
&s \mapsto \frac{1}{\sqrt{2}}\left(\sqrt{1+\alpha_3(r)}, \frac{\alpha_2(r) - i\alpha_1(r)}{\sqrt{1+\alpha_3(r)}}\right)e^{2\pi i s + i\theta_2(r)}, \\
&r := 4\pi ts - \lfloor 4\pi ts \rfloor
\end{aligned} \tag{2}$$

for $d = 3$ and $C \geq 2\pi\sqrt{32t^4 - 8t^2 + 1}$ (with equality when θ_1 is constant for some θ_2).

By Proposition 2.3 (which communicates an analogue of a result [12, Theorem 1.1] of Ehler and Gröchenig in the weighted setting), the weighted t-design curves described in Theorem 1.3 are *asymptotically optimal*, in the sense that any sequence $\{w_t\}_{t=0}^{\infty}$ of such curves on S^d achieves the optimal asymptotic order of length $\ell(w_t) \asymp t^{d-1}$ as $t \to \infty$ of a sequence indexed in t of weighted t-design curves on S^d.

In analogy with results concerning the asymptotic order of size of weighted t-design sets on spheres [31, 10], we also prove a result concerning the asymptotic order of length of weighted t-design curves in dimension d for fixed strength t.

Theorem 1.4. *For any $t \in \mathbb{N}$, there exists a constant $D_t \in 2\mathbb{Z}$ such that for any $d \in \mathbb{N}_+$ and $D \geq \pi D_t d^{t-1}$, there exists a weighted t-design curve on S^d of length D.*

Note that the weighted t-design curves said to exist in Theorem 1.4 are not in general *asymptotically optimal as* $d \to \infty$: we may produce weighted 2-design and 3-design curves on S^d of lengths πd and $2\pi d$ respectively by applying Theorem 1.5 to the vertices of a regular simplex (for d even) and cross-polytope on S^d, which respectively constitute 2-design and 3-design sets. Theorem 1.5 communicates a construction of weighted t-design curves on S^d from weighted t-design sets on S^{d-1} which we apply alongside existence results for unweighted and weighted t-design sets satisfying the desired asymptotics in strength t [4, Theorem 1] and in dimension d [10, Theorem 1.6] to prove Theorems 1.3 and 1.4.

Theorem 1.5. *For $d > 1$, $t \in \mathbb{N}$, a weighted t-design set $(X = \{x_i\}_{i=1}^{2N}, \lambda)$ on S^{d-1}, and any smooth path $M : [-1, 1] \to O(d)$ in the space of orthogonal transformations of $\mathbb{R}^d$, $w_{X,M}$ constructed as in* (3) *is a weighted t-design curve on S^d of length greater than or equal to $\pi|X|$ (with equality when M is constant) with self-intersections and singularities exactly in the set $\{\sum_{j=0}^{i} \lambda(x_j)\}_{i=0}^{|X|}$.*

We now discuss the main results of this work in the setting of approximate t-design curves.

Theorem 1.6 (Main theorem on approximate t-design curves)**.** *For any $n \in \mathbb{N}$, there exists a sequence $(\gamma_t)_{t=0}^{\infty}$ of simple ε_t-approximate t-design cycles on S^{2n+1} satisfying $\ell(\gamma_t) \asymp t^{2n}$ and $\varepsilon_t \asymp 1/t$ as $t \to \infty$.*

From Theorem 1.6, we see Theorem 1.7 by applying a construction [12, Sections 4-6] of Ehler and Gröchenig that builds a t-design curve on S^d by placing a t-design curve on each spherical cap around a point of a t-design set on S^d which Proposition 5.4 communicates in the approximate setting.

Theorem 1.7. *For any $n \in \mathbb{N}$, there exists a sequence $(\gamma_t)_{t=0}^{\infty}$ of simple ε_t-approximate t-design curves on S^{2n} satisfying $\varepsilon_t \asymp 1/t$ and $\ell(\gamma_t) \asymp t^{4n-3}$ as $t \to \infty$.*

We discuss in Remark 3.5 how, for any $p \in \mathbb{N}$, the sequences discussed in Theorems 1.6 and 1.7 are also sequences of $(\varepsilon_t, ||\cdot||_p)$-approximate t-design curves, where such an approximate design curve is a curve satisfying a condition equivalent to Definition 1.2 but where the L^p norm $||\cdot||_p$ is used instead of the L^∞ (supremum) norm. Thus, the subsequence $(\gamma_{pt})_{t=0}^{\infty}$ for γ_t as in either theorem satisfies the definition [13, Definition 4.1] of a Marcinkiewicz-Zygmund family for L^q for all even $q \in \mathbb{N}$ in addition to satisfying the definition for $q = \infty$.

We prove Theorem 1.6 by combining existence results for t-design sets on the complex projective space $\mathbb{CP}^n$ we define in Subsection 5.1 satisfying the desired asymptotics in strength t with a construction of approximate

t-design curves on S^{2n+1} from $\lfloor t/2 \rfloor$-design sets on $\mathbb{CP}^n$ communicated by Theorem 1.8.

Theorem 1.8. *Fix $n, t \in \mathbb{N}$ alongside a $\lfloor t/2 \rfloor$-design set Y on $\mathbb{CP}^n$ and constants*

$$W_Y \in \left(0, (|Y| - 1) \sup_{y_1, y_2 \in Y} d(y_1, y_2)\right], \quad M_Y > 0$$

as in Subsection 4.2. For any $\delta \in (0, M_Y)$, we may construct a simple $((W_Y + \delta)/(2\pi|Y|), 1/(2\pi|Y|))$-approximate t-design cycle γ_Y on S^{2n+1} of length $\ell(\gamma_Y) = 2\pi|Y| + W_Y - \delta$.

Applying Remark 3.1, we can see that γ_Y as in Theorem 1.8 will also be a $(W_Y/(\pi|Y|))$-approximate t-design curve. Additionally, applying Lemma 3.3 alongside Remark 3.1 and noting that we can make $\delta > 0$ arbitrarily small, we can see that Theorem 1.8 still holds when γ_Y is a smooth ε_Y-approximate t-design curve rather than an ε_Y-approximate t-design cycle. Similarly, Theorem 1.6 also holds when $(\gamma_t)_{t=0}^{\infty}$ is a sequence of smooth ε_t-approximate t-design curves rather than ε_t-approximate t-design cycles.

Note that Theorems 1.6 and 1.8 hold (with slight changes to constants in Theorem 1.8 by a factor of (about) 2) when S^d is replaced with d-dimensional real projective space $\mathbb{RP}^d$. We do not explicitly treat the real projective case to slightly simplify the proofs below, but the theorems follow from exactly the same methods described.

We prove Theorem 1.5 and then apply this theorem to show Theorems 1.3 and 1.4 in Section 2. We then make basic observations about approximate t-design curves in Section 3 and apply these observations and other results to prove Theorem 1.8 in Section 4 and Theorem 1.6 in Section 5. Finally, we discuss further related work in Section 6.

2. Weighted t-design curves

In this section, we prove Theorem 1.5, then use this result combined with existence results for unweighted and weighted t-design sets satisfying the desired asymptotics in strength t [4, Theorem 1] and in dimension d [10, Theorem 1.6] to prove Theorems 1.3 and 1.4. To this end, we first introduce weighted t-design sets.

Definition 2.1. The pair $(X, \lambda : X \to \mathbb{R})$ is called a *weighted t-design set* if

$$\sum_{x \in X} \lambda(x) f(x) = \int_{S^d} f \, d\sigma$$

for all $f \in P_t(S^d)$.

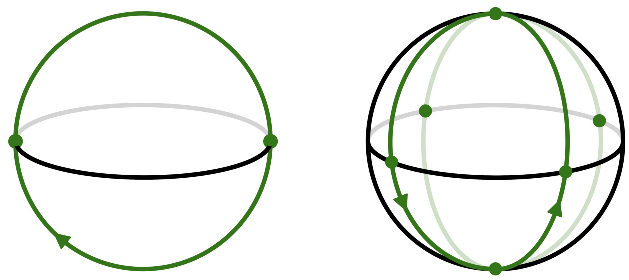

FIGURE 1. Images of the weighted 1-design curve $w_{V_2,\mathrm{Id}}$ (left) and the weighted 3-design curve $w_{V_4,\mathrm{Id}}$ (right) on S^2 respectively resulting from the construction of Theorem 1.5 applied to a 1-design set (antipodal points V_2) and 3-design set (the vertices V_4 of a 4-gon) on S^1.

For $d \in \mathbb{N}_+$, $t \in \mathbb{N}$, a weighted t-design set $(X = \{x_i\}_{i=1}^{2N}, \lambda)$ on S^{d-1}, and any smooth path $M : [-1, 1] \to O(d)$ in the space of orthogonal transformations of $\mathbb{R}^d$, we consider the curve

$$\begin{gathered} w_{X,M} = (w_{\mathbb{R}}, w_{\mathbb{R}^d}) : [0,1] \to S^d \subset \mathbb{R} \times \mathbb{R}^d, \\ w_{\mathbb{R}}(s) = (-1)^i(2(s - \Lambda_{i-1}) - 1), \quad w_{\mathbb{R}^d}(s) = \sqrt{1 - w_{\mathbb{R}}(s)^2} M(w_{\mathbb{R}}(s)) x_i \\ \textit{for} \quad i \in \{1, ..., 2N\}, \quad \Lambda_i := \sum_{j=1}^{i} \lambda(x_j), \quad s \in [\Lambda_{i-1}, \Lambda_i]. \end{gathered} \tag{3}$$

We may observe $w_{X,M}$ is a continuous, piecewise smooth curve of length at least $2\pi N$ (specifically, for any $\delta_M \in [0, \infty)$, we may select M such that $w_{X,M}$ has length $2\pi N + \delta_M$) which has self-intersections and singularities exactly at $s \in \{\Lambda_i\}_{i=0}^{2N}$. Additionally, note that $w_{X,M}$ is a geodesic cycle when M is constant. We will now verify that $w_{X,M}$ is a weighted t-design curve on S^d.

Proof of Theorem 1.5. Consider $d \in \mathbb{N}_+$, $t \in \mathbb{N}$, $w_{X,M}$ as in (3) and $f \in P_t(S^d)$ alongside the map $h : S^d \to [-1, 1]$ taking $\omega \mapsto \omega_1$, whose preimages $h^{-1}(s)$ we equip with the uniform spherical measure σ normalized so that $\sigma(h^{-1}(s)) = 1$. Note that $f|_{h^{-1}(s)}$ is a polynomial of degree at most t on

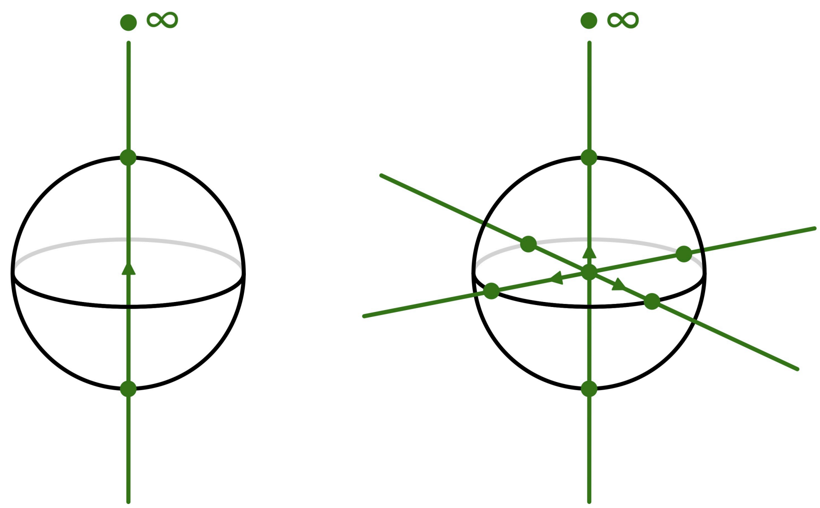

Figure 2. Images of a weighted 1-design curve (left) and a weighted 3-design curve (right) on $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ respectively resulting from the construction of Theorem 1.5 applied to a 1-design set (antipodal points) and 3-design set (the vertices of an octahedron) on S^2.

$h^{-1}(s) \cong S^{d-1}$ for all $s \in [0,1]$. We therefore see that

$$\begin{aligned}\int_0^1 f(w_{X,M}(s))\,ds &= \sum_{i=1}^{2N} \int_{\Lambda_{i-1}}^{\Lambda_i} f(w_{X,M}(s))\,ds \\ &= \sum_{i=1}^{2N} \frac{\lambda(x_i)}{2} \int_{-1}^{1} f(w_{X,M}(w_{\mathbb{R}}^{-1}(s)))\,ds \\ &= \frac{1}{2}\int_{-1}^{1} \sum_{i=1}^{2N} \lambda(x_i) f\left(s, \sqrt{1-s^2}M(s)x_i\right)\,ds \\ &= \frac{1}{2}\int_{-1}^{1}\int_{h^{-1}(s)} f(\omega)\,d\sigma(\omega)\,ds \\ &= \int_{S^d} f(\omega)\,d\sigma(\omega),\end{aligned}$$

so $w_{X,M}$ is a weighted t-design curve. □

Fix any $t \in \mathbb{N}$ and a continuous, piecewise smooth function $\theta_1 : [-1,1] \to \mathbb{R}$. Noting that the vertices V_{t+1} of a $(t+1)$-gon are a t-design set on S^1 for all $t \in \mathbb{N}$, we can see from Theorem 1.5 that $w_{V_{2t}, e^{i\theta_1(s)}}$ as in (3) is a weighted $(2t-1)$-design curve on S^2 of length greater than or equal to $2\pi t$ (with equality if θ_1 is constant). Observing that $w_{V_{2t}, e^{i\theta(s)}}$ is exactly the curve (1) then completes the proof of Theorem 1.3 when $d = 2$. Now, also note that the construction [22, Theorem 1.3] which builds a simple t-design curve on S^3 from a $\lfloor t/2 \rfloor$-design curve on S^2 directly generalizes to the setting of weighted t-design curves to provide the result of Theorem 2.2.

Theorem 2.2 (Weighted analogue of Theorem 1.3 in work [22] of the present author)**.** *For any $t \in \mathbb{N}$, a weighted $\lfloor t/2 \rfloor$-design curve $\alpha = (\alpha_{\mathbb{R}}, \alpha_{\mathbb{C}})$ on $S^2 \subset \mathbb{R} \times \mathbb{C}$ satisfying $\alpha_{\mathbb{R}}(s) \neq -1$ for $s \in [0,1]$, and any continuous, piecewise smooth function $\theta : [0,1] \to \mathbb{R}$ satisfying $\theta(0) - \theta(1) \in 2\pi\mathbb{Z}$, the curve*

$$\begin{gathered} \gamma_{\alpha,\theta} : [0,1] \to S^3 \subset \mathbb{C}^2, \\ s \mapsto \frac{1}{\sqrt{2}} \left(\sqrt{1 + \alpha_{\mathbb{R}}(r)}, \frac{\overline{\alpha_{\mathbb{C}}(r)}}{\sqrt{1 + \alpha_{\mathbb{R}}(r)}} \right) e^{2\pi i s + i\theta(r)} \\ \text{for} \quad r := (t+1)s - \lfloor (t+1)s \rfloor \end{gathered} \tag{4}$$

is a weighted t-design curve on S^3.

The proof of Theorem 2.2 follows as in the unweighted setting [22, Theorem 1.3], so we refer to that proof to verify the theorem. Note also that we may pick θ such that $\gamma_{\alpha,\theta}$ is an (unweighted) t-design curve (which we may assume is simple as in work of the present author [22, Proof of Theorem 1.3]—more generally, for any $\varepsilon > 0$ and θ_1, we may pick θ_2 such that γ_{α,θ_2} is simple and $\ell(\gamma_{\alpha,\theta_2}) = \ell(\gamma_{\alpha,\theta_1}) + \varepsilon$) when $|\alpha'|$ is bounded on the subset of $[0,1]$ on which α is smooth by varying θ such that $|\gamma'_{\alpha,\theta}(s)|$ is constant for all $s \in [0,1]$ at which $\gamma_{\alpha,\theta}$ is smooth.

Now, assuming $\theta_1(0) \neq m\pi - 2n\pi/t$ for all $m, n \in \mathbb{Z}$, we may observe that setting $(\alpha_1, \alpha_2, \alpha_3) = w_{V_{2t}, e^{i\theta(s)}}$, $\alpha_3(s) \neq -1$ for all $s \in [0,1]$. Considering a continuous, piecewise smooth curve $\theta_2 : [0,1] \to \mathbb{R}$ such that $\theta_2(0) - \theta_2(1) \in 2\pi\mathbb{Z}$, we may therefore set $\alpha = (\alpha_3, \alpha_2, \alpha_1)$ and $\theta = \theta_2$ in Theorem 1.3 to produce a weighted $(4t-1)$-design curve $\gamma_{(\alpha_3,\alpha_2,\alpha_1),\theta_2}$ on S^3 of length greater than or equal to $2\pi\sqrt{32t^4 - 8t^2 + 1}$ (with equality when θ_1 is constant for some θ_2). Also observe from Theorem 2.2 that for any θ_1 and θ_2, we may perturb θ_2 such that (2) will be simple. Then, since $\gamma_{(\alpha_3,\alpha_2,\alpha_1),\theta_2}$ is exactly the curve (2), we have completed the proof of Theorem 1.3 when $d = 3$.

We now prove Theorem 1.3 for any $d \in \mathbb{N}_+$. Take C_{d-1} as in work of Bondarenko, Radchenko, and Viazovska [4, Theorem 1] and add 1 to C_{d-1} if needed to assume it is even. Then, for any $t \in \mathbb{N}$, there exists a t-design set X on S^{d-1} of even size $C_{d-1}t^{d-1}$. For any $C \geq \pi C_{d-1}t^{d-1}$, we may therefore pick smooth $M : [0,1] \to O(d)$ such that $w_{X,M}$ is a weighted t-design curve of length C with self-intersections and singularities exactly in the set $\{i/C_{d-1}t^{d-1}\}_{i=0}^{C_{d-1}t^{d-1}}$, proving Theorem 1.3. We see from Proposition 2.3 that the asymptotic order of length of these curves is optimal among weighted t-design curves.

Proposition 2.3. *For each $d \in \mathbb{N}$, there exists a constant $c_d > 0$ such that for any $t \in \mathbb{N}$, the length of any weighted t-design curve on S^d is at least $c_d t^{d-1}$.*

This proposition follows exactly as the proof of Theorem 1.1 of Ehler and Gröchenig [12], which provides this result in the case of unweighted t-design curves.

Additionally, note from work of Dillon [10, Theorem 1.6] that for each $t \in \mathbb{N}$, there exists a constant D_t we may assume is even such that for any $d \in \mathbb{N}_+$, there exists a weighted t-design set on S^d of size at most $D_t d^{t-1}$. Then, for any $D > \pi D_t t^d$, we may pick smooth $M : [0,1] \to O(d)$ such that $w_{X,M}$ is a weighted t-design curve of length D with self-intersections and singularities at exactly $|X| + 1$ values in $[0, 1]$, proving Theorem 1.4.

3. Approximate t-design curves

We now prove facts about (ε, c)-approximate t-design curves. We first discuss basic observations about (ε, c)-approximate t-design curves in Subsection 3.1, then discuss how (ε, c)-approximate t-design curves satisfy the analogue of Definition 1.2 in which the supremum of f on S^d is replaced with the L^p norm of f on S^d for all $p \in \mathbb{N}_+$ in Subsection 3.2.

3.1. Initial observations.

To the end of making initial observations about (ε, c)-approximate t-design curves on S^d, fix any $d \in \mathbb{N}_+$, $t \in \mathbb{N}$, $\varepsilon \geq 0$, and $c > 0$. The first observation we find it natural to make about (ε, c)-approximate t-design curves is that any continuous, piecewise smooth curve $\gamma : [0,1] \to S^d$ is a $(c\ell(\gamma) + 1, c)$-approximate t-design curve on S^d for any $c > 0$, while 0-approximate t-design curves are exactly t-design curves as defined by Ehler and Gröchenig [12, Definition 2.1]. Additionally, we may observe taking $f = 1$ in Definition 1.2 that there are no (ε, c)-approximate t-design curves γ for $c \notin [(1-\varepsilon)/\ell(\gamma), (1+\varepsilon)/\ell(\gamma)]$.

We now discuss how the error term ε changes when we change the scaling constant c associated to an (ε, c)-approximate t-design curve, which we use in Subsection 5.2 to assume the curves $(\gamma_t)_{t=0}^{\infty}$ in Theorem 1.6 have scaling constant $1/\ell(\gamma_t)$; i.e. that they are ε-approximate t-design curves.

Remark 3.1. Consider any $\widetilde{c} > 0$ alongside an (ε, c)-approximate t-design curve γ on S^d. We may see since

$$\left| \widetilde{c} \int_\gamma f - \int_{S^d} f \, d\sigma \right| \leq \varepsilon \left| \sup_{S^d} f \right| + \left| (\widetilde{c} - c) \int_\gamma f \right| = (\varepsilon + |\widetilde{c} - c| \ell(\gamma)) \left| \sup_{S^d} f \right|$$

for any $f \in P_t(S^d)$ that γ is also an $(\varepsilon + |\widetilde{c} - c|\ell(\gamma), \widetilde{c})$-approximate t-design curve.

We now mention how a continuous, piecewise smooth, closed curve whose image coincides with a piecewise smooth topological 1-manifold which exactly averages polynomials of degree at most t on S^d will be an (ε, c)-approximate t-design curve for ε and c which depend on how closely the image of the curve coincides with the manifold.

Lemma 3.2. *Consider a piecewise smooth topological 1-manifold $L \subset S^d$ satisfying*

$$\left| c \int_L f \, ds - \int_{S^d} f \, d\sigma \right| \leq \varepsilon \left| \sup_{S^d} f \right|$$

and a continuous, piecewise smooth, closed curve $\gamma : [0,1] \to S^d$, where s is the arc length measure on L. Defining $l := \gamma^{-1}(\gamma([0,1]) \cap L)$ and setting $\varepsilon_\gamma := \varepsilon + c(|L| + \ell(\gamma) - 2\ell(\gamma|_l))$ for $|L| := s(L)$, γ is an (ε_γ, c)-approximate t-design curve.

Proof. For any $f \in P_t(S^d)$ and taking s to be the arc length measure on L, we have

$$\begin{aligned} \int_L f \, ds &= \int_{\gamma|_l} f + \int_{L \setminus \gamma(l)} f \, ds + \int_{\gamma|_{[0,1]\setminus l}} f - \int_{\gamma|_{[0,1]\setminus l}} f \\ &= \int_\gamma f + \int_{L \setminus \gamma(l)} f \, ds - \int_{\gamma|_{[0,1]\setminus l}} f. \end{aligned}$$

Therefore, we can see that

$$\begin{aligned} \varepsilon \left| \sup_{S^d} f \right| &\geq \left| c \int_L f \, ds - \int_{S^d} f \, d\sigma \right| \\ &= \left| c \left(\int_\gamma f + \int_{L \setminus \gamma(l)} f \, ds - \int_{\gamma|_{[0,1]\setminus l}} f \right) - \int_{S^d} f \, d\sigma \right| \\ &\geq \left| \left| c \int_\gamma f - \int_{S^d} f \, d\sigma \right| - c \left| \int_{L \setminus \gamma(l)} f \, ds - \int_{\gamma|_{[0,1]\setminus l}} f \right| \right|, \end{aligned}$$

so we have

$$\begin{aligned} \left| c \int_\gamma f - \int_{S^d} f \, d\sigma \right| &\leq \varepsilon \left| \sup_{S^d} f \right| + c \left| \int_{L \setminus \gamma(l)} f \, ds - \int_{\gamma|_{[0,1]\setminus l}} f \right| \\ &\leq (\varepsilon + c(|L| + \ell(\gamma) - 2\ell(\gamma|_l))) \left| \sup_{S^d} f \right|. \end{aligned}$$

This is exactly the desired result. □

We now discuss how any (ε, c)-approximate t-design curve on S^d may be perturbed to be smooth and, if $d \neq 2$, simple.

Lemma 3.3. *Consider an (ε, c)-approximate t-design curve γ on S^d. For any $\delta > 0$, there exists a smooth $(\varepsilon + 2\delta c, c)$-approximate t-design curve $\widetilde{\gamma}$ on S^d of length $\ell(\widetilde{\gamma}) = \ell(\gamma) + \delta$. If $d \neq 2$, we may take $\widetilde{\gamma}$ to be simple.*

Proof. With notation as in the theorem statement, take S to be the (finite) set of $s \in [0,1]$ such that $\gamma(s)$ is not smooth. Consider an open subset $\widetilde{S} \subset [0,1]$ such that $\ell(\gamma|_S) = \delta/2$ and $S \subset \gamma(S)$. We may then smooth γ to any curve $\widetilde{\gamma}$ which coincides with γ on $[0,1] \setminus S$ and satisfies $\ell(\widetilde{\gamma}|_S) = \delta/2$. We then see from Lemma 3.2 that $\widetilde{\gamma}$ satisfies the desired properties. As $(\varepsilon + 2\delta c, c)$-approximate t-design curves have finitely many self-intersection points, we may perform this same procedure taking S to be these self-intersection points to ensure $\widetilde{\gamma}$ will be simple. □

3.2. L^p **norms.** For a norm $||\cdot||_*$ on the space $P_t(S^d)$ of polynomials on S^d, we say a continuous, piecewise smooth, closed curve $\gamma : [0,1] \to S^d$ with finitely many self-intersections is an $(\varepsilon, c, ||\cdot||_*)$-approximate t-design curve if

$$\left| c \int_\gamma f - \int_{S^d} f \, d\sigma \right| \leq \varepsilon ||f||_*$$

for all $f \in P_t(S^d)$. Fix norms $||\cdot||_1$ and $||\cdot||_2$ on $P_t(S^d)$. As $P_t(S^d)$ is a finite-dimensional vector space, there exist constants $c_t, C_t > 0$ such that

$$c_t ||f||_1 \leq ||f||_2 \leq C_t ||f||_1$$

for all $f \in P_t(S^d)$. Then, we see that any $(\varepsilon, c, ||\cdot||_2)$-approximate t-design curve is a $(C_t\varepsilon, c, ||\cdot||_1)$-approximate t-design curve. Therefore, taking $||\cdot||_1 := ||\cdot||_\infty$ to be the L^∞ norm on $P_t(S^d)$, we see that an $(\varepsilon, c, ||\cdot||_2)$-approximate t-design curve is a $(C_t\varepsilon, c)$-approximate t-design curve. The curves γ_t of interest in Theorems 1.6 and 1.7 are then $(C_t\varepsilon_t, c)$-approximate t-design curves. Taking $||\cdot||_2 = ||\cdot||_p$ to be the L^p norm on $P_t(S^d)$ for any $p \in \mathbb{N}$ and recalling that we normalize the uniform spherical measure σ on S^d such that $\sigma(S^d) = 1$, we then see that

$$||f||_p = \left(\int_{S^d} \left| \sup_{S^d} f \right|^p d\sigma \right)^{1/p} \leq \left| \sup_{S^d} f \right| = ||f||_\infty,$$

so we may take $C_t := 1$ to prove Remark 3.4.

Remark 3.4 (Corollary of Theorem 1.6)**.** For any $d \in \mathbb{N}$ and $\varepsilon > 0$, any ε-approximate t-design cycle on S^d is a $(\varepsilon, ||\cdot||_p)$-approximate t-design cycle on S^d for all $p \in \mathbb{N}$.

Remark 3.5. For any $p \in \mathbb{N}$, a sequence $(\gamma_t)_{t=0}^\infty$ as discussed in Theorems 1.6 or 1.7 is thus a sequence of $(\varepsilon_t, ||\cdot||_p)$-approximate t-design cycles for all $p \in \mathbb{N}$. Note then that for any $p \in \mathbb{N}_+$, the sequence $(\gamma_{pt})_{t=0}^\infty$ satisfies the definition [13, Definition 4.1] of a Marcinkiewicz-Zygmund family for $L^q(S^d)$ for all even $q \in \{\infty, 1, ..., p\}$ with $A = (1 - \widetilde{\varepsilon}_p)^{1/p}$ and $B = (1 + \widetilde{\varepsilon}_p)^{1/p}$, where we have

$$\widetilde{\varepsilon}_p := \sup_{t \in \mathbb{N}} \varepsilon_{pt} \asymp \frac{1}{p} \quad as \quad p \to \infty.$$

4. Building approximate t-design curves

In this section, we prove Theorem 1.8. We introduce complex projective t-design sets and discuss how they can be used to average degree at most $2t+1$ spherical polynomials in Subsection 4.1, then describe the construction of Theorem 1.8 and use the understanding of complex projective t-design sets discussed in Subsection 4.1 to verify the theorem in Subsection 4.2. As noted in Section 1, note this proof applies to show the result of Theorem 1.8 (with slight modifications to certain constants by a factor of (about) 2) when S^d is replaced by $\mathbb{RP}^d$.

4.1. Complex projective t-design sets.

In this subsection, we first introduce complex projective t-design sets, then discuss how these objects can be used to average degree at most $2t+1$ polynomials on spheres. To this end, we define the *complex projective space* $\mathbb{CP}^n$ of complex dimension n to be the quotient of S^{2n+1} by the multiplicative action $a \mapsto az$ of $S^1 \subset \mathbb{C}$ on $(n+1)$-dimensional complex vector space $\mathbb{C}^{n+1} \supset S^{2n+1}$ and equip $\mathbb{CP}^n$ with its uniform measure ρ normalized so that $\rho(\mathbb{CP}^n) = 1$. We also consider the *complex projective map*

$$\Pi : S^{2n+1} \to \mathbb{CP}^n, \quad \omega \mapsto [\omega] := \{\omega z \mid z \in S^1\} \tag{5}$$

(which we note pushes forward the uniform spherical measure σ to ρ) and denote by

$$P_t(\mathbb{CP}^n) := \{f : \mathbb{CP}^n \to \mathbb{R} \mid \Pi^* f \in P_{2t}(S^{2n+1})\} \tag{6}$$

the space of polynomials of degree at most t on $\mathbb{CP}^n$. We then say a finite subset $Y \subset \mathbb{CP}^n$ is a *complex projective t-design set* on $\mathbb{CP}^n$ if

$$\frac{1}{|Y|} \sum_{y \in Y} g(y) = \int_{\mathbb{CP}^n} g \, d\rho$$

for all $g \in P_t(\mathbb{CP}^n)$. Lemma 4.1 describes how complex projective t-design sets average spherical polynomials.

Lemma 4.1. *Consider a t-design set Y on $\mathbb{CP}^n$. For any $f \in P_{2t+1}(S^{2n+1})$, we have*

$$\frac{1}{2\pi|Y|} \int_{\Pi^{-1}(Y)} f \, ds = \int_{S^d} f \, d\sigma,$$

where s is the arc length measure on $\Pi^{-1}(Y)$.

This is the fundamental observation behind the construction which uses the complex projective map to build a family of t-design sets on S^{2n+1} from a $\lfloor t/2 \rfloor$-design set on $\mathbb{CP}^n$ introduced by König [18, Corollary 1] and also investigated by Kuperberg [19, Theorem 4.1]. This phenomenon was further investigated by Okuda [26, Theorem 1.1], who—inspired by the independent observation of the construction by Cohn, Conway, Elkies, and Kumar [6, Section 4]—verified a related construction which uses the Hopf map to build

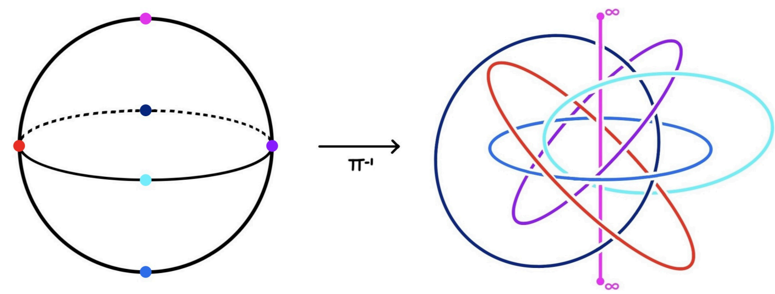

FIGURE 3. The preimage $\Pi^{-1}(O) \subset S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ under Π of the 3-design set on $\mathbb{CP}^1$ corresponding to the vertices O of an octahedron under the association $\mathbb{CP}^1 \cong S^2$ [3, Example 2.7]. The average of any degree at most 7 polynomial on $\Pi^{-1}(O)$ equals the average of this polynomial on S^3.

a family of t-design sets on S^3 from a $\lfloor t/2 \rfloor$-design set on $S^2 \cong \mathbb{CP}^1$. This was generalized to work of the present author [21, Theorem 1.1] which verifies constructions that build a t-design set on S^{4n+3} from a $\lfloor t/2 \rfloor$-design set on the quaternionic projective space $\mathbb{HP}^n$ (or $S^4 \cong \mathbb{HP}^1$ when $n = 1$) and a t-design set on S^3 and a t-design set on S^{15} from a $\lfloor t/2 \rfloor$-design set on the octonionic projective line $\mathbb{OP}^1$ (or $S^8 \cong \mathbb{OP}^1$) and a t-design set on S^7. The methods used throughout this work can be applied alongside the result analogous to Lemma 4.1 in these quaternionic and octonionic settings to give rise to an analogue of Theorem 1.8 where Y is a $\lfloor t/2 \rfloor$-design set on $\mathbb{HP}^n$ or $\mathbb{OP}^1 \cong S^8$ and γ_Y is diffeomorphic to S^3 or S^7 respectively and approximately averages degree at most t polynomials in a sense (13) similar to that of Definition 1.2.

A proof of Lemma 4.1 can be found by appealing to the result of König [18, Corollary 1] or by applying methods used in that work, the work of Kuperberg [19, Theorem 4.1], or that of the present author [21, Theorem 1.1].

4.2. **Connecting fibers.** In this subsection, we prove Lemma 4.2, then apply Lemma 3.2 to prove Theorem 1.8. To this end, for any finite subset $Y \subset \mathbb{CP}^n$, we consider a minimal spanning tree $\mathcal{T}_Y$ (which may be constructed using any of a number of simple algorithms [24]) for the graph with vertices the elements of Y and one edge for each pair $y_1, y_2 \in Y$ such that there exists a minimal geodesic between y_1 and y_2 whose interior is disjoint from Y which we assign a weighting equal to the length of this minimal geodesic measured with respect to the metric

$$d_{\mathbb{CP}^n}([\omega_1], [\omega_2]) := \inf_{z \in S^1} \arccos \langle \omega_1, z\omega_2 \rangle_{\mathbb{R}} \tag{7}$$

on $\mathbb{CP}^n$, where $\langle , \rangle_{\mathbb{R}}$ denotes the standard inner product on $\mathbb{R}^{2n+2}$. We then take W_Y to be double the sum of all weights of edges of $\mathcal{T}_Y$, N_Y

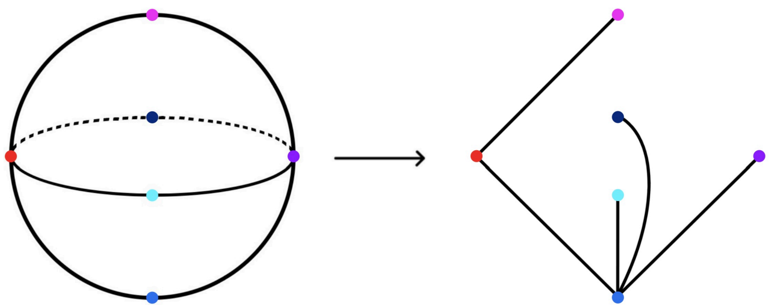

FIGURE 4. Making the tree $\mathcal{T}_O$ as in Section 4 associated to the vertices O of an octahedron on $\mathbb{CP}^1 \cong S^2$.

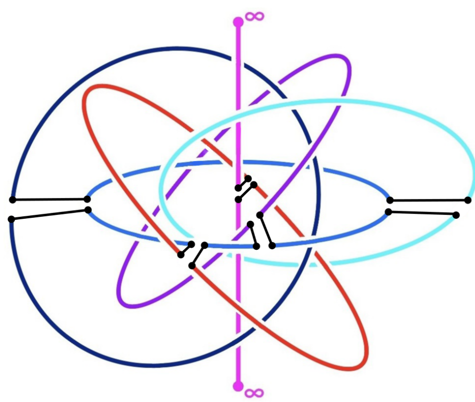

FIGURE 5. Connecting the disjoint circles comprising $\Pi^{-1}(O)$ to form a path-connected figure Γ_O in accordance with the tree $\mathcal{T}_O$ as in Figure 4. This is the image of a $((5\pi+\delta)/(12\pi-\delta), 12\pi/(17\pi-\delta))$-approximate 7-design cycle, with δ as in Theorem 1.8.

to be the maximal number of edges connected to any vertex of $\mathcal{T}_Y$, and $M_Y := 2\pi(|Y|-1)/N_Y$.

Lemma 4.2. *Consider $n \in \mathbb{N}$, a finite subset $Y \subset \mathbb{CP}^n$, any $\delta \in (0, M_Y)$, and the complex projective map $\Pi : S^{2n+1} \to \mathbb{CP}^n$ as in (5). We may construct a simple geodesic cycle γ_Y on S^{2n+1} satisfying*

$$|\gamma_Y([0,1]) \setminus (\gamma_Y([0,1]) \cap \Pi^{-1}(Y))| = W_Y,$$
$$|\Pi^{-1}(Y) \setminus (\gamma_Y([0,1]) \cap \Pi^{-1}(Y))| = \delta.$$

Applying Lemma 4.1 alongside Lemma 3.2 with $L := \Pi^{-1}(Y)$ and $\gamma := \gamma_Y$ for Y a complex projective $\lfloor t/2 \rfloor$-design set, we obtain Theorem 1.8 as a consequence of Lemma 4.2.

Proof of Lemma 4.2. Denote by $y_0 \in Y$ the root of the tree $\mathcal{T}_Y$. For each $y \in Y \setminus \{y_0\}$, consider the geodesic $e_y : [0,1] \to \mathbb{CP}^n$ in the set of edges of $\mathcal{T}_Y$ satisfying $e_y(0) = p$ (p the parent of y in $\mathcal{T}_Y$) and $e_y(1) = y$. For $z_y \in p$, we consider a geodesic $g_{z_y} : [0,1] \to S^3$ on S^3 satisfying $g_{z_y}(0) = z_y$ and $\Pi \circ g_y = e_y$ of minimal length, as can be explicitly constructed using a straightforward procedure which amounts to solving an ODE [22, Lemma 2.1].

Denoting by $\{d_{y,j}\}_{j=1}^{n_y}$ the $n_y \in \{0, ..., N_Y\}$ descendants of $y \in Y$ and considering any $\delta_{d_{y,j}} \in [0, 2\pi/N_Y - \widetilde{\delta})$ for $\widetilde{\delta} := \delta/(2(|Y| - 1))$, we pick any $z_{y_0} \in y_0 \subset S^3$ and iteratively define

$$z_{d_{y,j}} := g_{z_y}(1)e^{2\pi i j/n_y + \delta_{d_{y,j}} i} \in d_{y,j} \subset S^3$$

for all $y \in Y$ and $j \in \{1, ..., n_y\}$. We can observe that the sets

$$D_y := \{z_y e^{i\theta} \mid \theta \in [0, \widetilde{\delta}]\} \cup \{g_{z_y}(1)e^{i\theta} \mid \theta \in [0, \widetilde{\delta}]\} \subset \Pi^{-1}(Y)$$

are pairwise disjoint for all $y \in Y \setminus \{y_0\}$, as for any $y \in Y$ and $j \in \{1, ..., n_y\}$, we have

$$2\pi/n_y - \delta_{d_{y,j-1}} + \delta_{d_{y,j}} \geq 2\pi/n_y - \delta_{d_{y,j-1}} > 2\pi/N_Y - 2\pi/N_Y + \widetilde{\delta} = \widetilde{\delta}.$$

We can also arrange that the sets

$$G_y := g_{z_y}([0,1]) \cup \left(g_{z_y}([0,1])e^{i\widetilde{\delta}}\right)$$

are pairwise disjoint by perturbing $\delta_y \in [0, 2\pi/N_Y - \widetilde{\delta})$ for each y such that there exists $\widetilde{y} \in Y \setminus \{y_0\}$ satisfying $G_y \cap G_{\widetilde{y}} \neq \emptyset$. Therefore, noting that $G_y \cap \Pi^{-1}(Y) = \partial G_y = \partial D_y$ for all $y \in Y$ and that $\Pi^{-1}(Y)$ and G_y are finite unions of geodesics in S^{2n+1}, we can see that the set

$$\Gamma_Y := \left(\Pi^{-1}(Y) \setminus \bigcup_{y \in Y} D_y\right) \bigcup_{y \in Y} G_y$$

is a finite union of geodesics in S^{2n+1}. Observing that $\Pi(G_y) = e_y([0,1])$, we can see that $\Pi(\Gamma_Y)$ is exactly the union of the edges and vertices of the connected graph $\mathcal{T}_Y$ and thus is connected. Therefore, picking any continuous, piecewise smooth, simple curve $\gamma_Y : [0,1] \to \Gamma_Y$, γ_Y will be a geodesic cycle satisfying

$$|\gamma_Y([0,1]) \setminus (\gamma_Y([0,1]) \cap \Pi^{-1}(Y))| = \left|\bigcup_{y \in Y} G_y\right| = 2\sum_{y \in Y} \ell(g_y) = W_Y,$$

$$|\Pi^{-1}(Y) \setminus (\gamma_Y([0,1]) \cap \Pi^{-1}(Y))| = \left|\bigcup_{y \in Y} D_y\right| = 2(|Y| - 1)\widetilde{\delta} = \delta$$

as desired. □

5. Asymptotics of approximate t-design curves

We now use results concerning the asymptotic sizes of sequences of complex projective t-design sets stated in Subsection 5.1 alongside the construction communicated by Theorem 1.8 to prove Theorem 1.6 in Subsection 5.2, then apply the construction established by Ehler and Gröchenig that builds a t-design curve on S^d from a t-design curve on S^{d-1} (which extends to a construction of approximate t-design curves) to prove Theorem 1.7 in Subsection 5.3.

5.1. **Complex projective t-design sets.** In this subsection, we discuss asymptotic results concerning different notions of complex projective t-design sets, then present Lemma 5.3, which describes how these notions are equivalent. To this end, consider the embedding

$$\varphi : \mathbb{CP}^n \to \mathbb{C}^{(n+1)^2} \cong \mathbb{R}^{(2n+2)^2}, \quad [\omega] \mapsto \omega\omega^*$$

(ω^* the conjugate transpose of ω) of $\mathbb{CP}^n$ into Euclidean space taking a point $[\omega]$ the projection matrix mapping $\mathbb{C}^{n+1}$ to the complex line containing ω. We then define

$$Q_t(\mathbb{CP}^n) := \varphi^*(P_t(\mathbb{R}^{(2n+2)^2})). \tag{8}$$

Additionally, for $\Delta_{\mathbb{CP}^n}$ the Laplace-Beltrami operator on $\mathbb{CP}^n$, we denote by $R_t(\mathbb{CP}^n)$ the space of *diffusion polynomials* of degree at most t on $\mathbb{CP}^n$, the span over $\mathbb{R}$ of real-valued eigenfunctions of $\Delta_{\mathbb{CP}^n}$ in $L^1(\mathbb{CP}^n)$ of eigenvalue at most $4t(t+n)$. Then, taking S_t to be either Q_t or R_t, we say a finite subset $Y \subset \mathbb{CP}^n$ is a *S_t-design set* if

$$\frac{1}{|Y|}\sum_{y\in Y} g(y) = \int_{\mathbb{CP}^n} g\, d\rho$$

for all $g \in S_t(\mathbb{CP}^n)$.

Proposition 5.1 (The $\mathcal{M} = \mathbb{CP}^n$ case of Theorem 2.2 of Etayo, Marzo and Ortega-Cerdà [14])**.** *For any $n \in \mathbb{N}$, there exists a sequence $\{Y_t\}_{t=0}^\infty$ of Q_t-design sets on $\mathbb{CP}^n$ of asymptotically optimal size $|Y_t| \asymp t^{2n}$.*

Proposition 5.2 (Theorem 3.2 of Breger, Ehler, and Gräf [5])**.** *Fix $n \in \mathbb{N}$ and consider a sequence $\{Y_t\}_{t=0}^\infty$ of R_t-design sets on $\mathbb{CP}^n$. Defining*

$$D_t := \sup_{y_1\in Y_t} \inf_{y_2\in Y_t\setminus\{y_1\}} d_{\mathbb{CP}^n}(y_1, y_2)$$

for $d_{\mathbb{CP}^n}$ the metric on $\mathbb{CP}^n$ as in (7)*, we have $D_t \asymp 1/t$ as $t \to \infty$.*

We use Lemma 5.3 to apply Propositions 5.1 and 5.2 to the setting of t-design sets. For completeness, we provide a full proof of the lemma.

Lemma 5.3. *For any $t \in \mathbb{N}$, we have $P_t(\mathbb{CP}^n) = Q_t(\mathbb{CP}^n) = R_t(\mathbb{CP}^n)$.*

As a corollary of this lemma, we see that the conditions that a finite subset of $\mathbb{CP}^n$ is a t-design set, a Q_t-design set, and a R_t-design set are equivalent.

Proof of Lemma 5.3. We first show that $P_t(\mathbb{CP}^n) = Q_t(\mathbb{CP}^n)$. To this end, note that for any $c \in \mathbb{C}$, the real and imaginary parts of c respectively satisfy

$$\Re(c) = \frac{1}{2}(\overline{c} + c), \quad \Im(c) = \frac{1}{2}(\overline{c} - c)i. \tag{9}$$

Observe from the definition (8) of $Q_t(\mathbb{CP}^n)$ that $\Pi^*(Q_t(\mathbb{CP}^n))$ is exactly the space of restrictions to S^{2n+1} of elements in the span over $\mathbb{R}$ of products of at most t of the functions

$$\omega \mapsto \Re(\omega_i \overline{\omega_j}), \quad \omega \mapsto \Im(\omega_i \overline{\omega_j}) \quad (i, j \in \{1, ..., n+1\})$$

taking $\omega \in \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ to the real and imaginary parts of entries of $\omega\omega^*$. Thus, we may see from (9) that $\Pi^*(Q_t(\mathbb{CP}^n)) \subset P_{2t}(S^{2n+1})$, so $Q_t(\mathbb{CP}^n) \subset P_t(\mathbb{CP}^n)$. To the end of showing the reverse inclusion, pick $g \in P_t(\mathbb{CP}^n)$. We see from the definition (6) of $P_t(\mathbb{CP}^n)$ that $\Pi^* g$ is in the space of restrictions to S^{2n+1} of elements in the span over $\mathbb{R}$ of products of at most $2t$ of the functions

$$\omega \mapsto \Re(\omega_i), \quad \omega \mapsto \Im(\omega_i) \quad (i \in \{1, ..., n+1\})$$

taking $\omega \in \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ to the real and imaginary parts of its entries. (9) allows us to pick $c_{\alpha,\beta} \in \mathbb{C}$ for each $(\alpha, \beta) \in I$ such that

$$(\Pi^* g)(\omega) = \sum_{(\alpha,\beta) \in I} c_{\alpha,\beta} \omega^{\alpha,\beta} \quad \textit{for} \quad \omega^{\alpha,\beta} := \prod_{i=1}^{n+1} \omega_i^{\alpha_i} \overline{\omega_i}^{\beta_i},$$

where we define

$$I := \left\{ (\alpha, \beta) \in \mathbb{N}^{2n+2} \;\middle|\; \sum_{i=1}^{n+1} (\alpha_i + \beta_i) \leq 2t \right\},$$
$$I_0 := \left\{ (\alpha, \beta) \in I \;\middle|\; \sum_{i=1}^{n+1} (\alpha_i - \beta_i) = 0 \right\}.$$

Note that $\omega^{\alpha,\beta} \in Q_t(\mathbb{CP}^n)$ for any $(\alpha, \beta) \in I_0$, so if we show that

$$(\Pi^* g)(\omega) = \sum_{(\alpha,\beta) \in I_0} c_{\alpha,\beta} \omega^{\alpha,\beta}, \tag{10}$$

we will have $\Pi^* g \in Q_t(\mathbb{CP}^n)$. Picking any $z_w = (z_{w,1}, ..., z_{w,n+1}) \in w$ for $w \in \mathbb{CP}^n$ and fixing any $(a, b) \in I$, we may observe that

$$\begin{aligned}
\int_w \omega^{\alpha,\beta}\, d\sigma(\omega) &= \int_{S^1} \prod_{i=1}^{n+1} (z_{w,i}\zeta)^{\alpha_i} (\overline{z_{w,i}\zeta})^{\beta_i}\, d\sigma(\zeta) \\
&= \left(\prod_{i=1}^{n+1} z_{w,i}^{\alpha_i} \overline{z_{w,i}}^{\beta_i} \right) \int_{S^1} \prod_{i=1}^{n+1} \zeta^{\alpha_i} \overline{\zeta}^{\beta_i}\, d\sigma(\zeta) \\
&= \left(\prod_{i=1}^{n+1} z_{w,i}^{\alpha_i} \overline{z_{w,i}}^{\beta_i} \right) \int_{S^1} \zeta^{\sum_{i=1}^{n+1} (\alpha_i - \beta_i)}\, d\sigma(\zeta),
\end{aligned}$$

which equals 0 whenever $(\alpha, \beta) \notin I_0$. Picking any $\omega \in w \in \mathbb{CP}^n$ and observing that $\omega^{\alpha,\beta} \in \Pi^*(L^1(\mathbb{CP}^n))$ when $(\alpha, \beta) \in I_0$ and that

$$f(\omega) = \int_w f\, d\sigma$$

for any $f \in \Pi^*(L^1(\mathbb{CP}^n))$, we therefore have that

$$\begin{aligned}
(\Pi^* g)(\omega) &= \int_w \Pi^* g\, d\sigma \\
&= \sum_{(\alpha,\beta) \in I} c_{\alpha,\beta} \int_w \omega^{\alpha,\beta}\, d\sigma(\omega) \\
&= \sum_{(\alpha,\beta) \in I_0} c_{\alpha,\beta} \int_w \omega^{\alpha,\beta}\, d\sigma(\omega) \\
&= \sum_{(\alpha,\beta) \in I_0} c_{\alpha,\beta} \omega^{\alpha,\beta},
\end{aligned}$$

showing (10). Thus, $\Pi^* g \in \Pi^*(Q_t(\mathbb{CP}^n))$, so $P_t(\mathbb{CP}^n) \subset Q_t(\mathbb{CP}^n)$. This completes the proof that $P_t(\mathbb{CP}^n) = Q_t(\mathbb{CP}^n)$.

We now show that $P_t(\mathbb{CP}^n) = R_t(\mathbb{CP}^n)$. To this end, note that for M an m-dimensional Riemannian manifold with Riemannian metric $\langle , \rangle$, geodesic normal coordinates $\{x_i : M \to \mathbb{R}\}_{i=1}^m$ in a neighborhood U of $p \in M$, and associated local frame $\{\partial_i := \frac{\partial}{\partial x_i}\}_{i=1}^m$, we have $\langle \partial_i, \partial_j \rangle = \delta_{i,j}$ ($\delta_{i,j}$ the Kronecker delta), so for any differentiable function $f : M \to \mathbb{R}$, the gradient ∇f of f satisfies

$$\nabla f = \sum_{i,j=1}^m \delta_{ij} \partial_j(f) \partial_i = \sum_{j=1}^m \partial_j(f) \partial_i$$

in U. Additionally noting that $(\nabla_{\partial_i}\partial_j)_p = 0$ for all $i, j \in \{1, ..., m\}$, we may see that for Δ the Laplace-Beltrami operator on M,

$$\begin{aligned}(\Delta f)(p) &= \sum_{i=1}^{m} \langle \nabla_{\partial_i} \nabla f, \partial_i \rangle_p \\ &= \sum_{i=1}^{m} \left\langle \nabla_{\partial_i} \sum_{j=1}^{m} \partial_j(f)\partial_j, \partial_i \right\rangle_p \\ &= \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} (\partial_i(\partial_j(f))\partial_j + \partial_j(f)\nabla_{\partial_i}\partial_j), \partial_i \right\rangle_p \\ &= \sum_{i=1}^{m} (\partial_i(\partial_i(f)))(p).\end{aligned} \tag{11}$$

Now, fix any $\omega \in w \in \mathbb{CP}^n$ and consider geodesic normal coordinates $\{x_i^{\mathbb{C}} : \mathbb{CP}^n \to \mathbb{R}\}_{i=1}^{2n}$ on $\mathbb{CP}^n$ in a neighborhood of w and the associated local frame $\{\partial_i^{\mathbb{C}} := \frac{\partial}{\partial x_i^{\mathbb{C}}}\}_{i=1}^{2n}$. Setting $x_i := \Pi^* x_i^{\mathbb{C}}$, we may then consider geodesic normal coordinates $\{x_i : S^{2n+1} \to \mathbb{R}\}_{i=1}^{2n+1}$ on S^{2n+1} in a neighborhood U of ω satisfying

$$\partial_{2n+1} = i\omega \in T_\omega S^{2n+1} \subset \mathbb{C}^{n+1}$$

for $\{\partial_i := \frac{\partial}{\partial x_i}\}_{i=1}^{2n+1}$ the associated local frame. We may observe from the definition of a local frame associated to a choice of local coordinates that $\partial_i \circ \Pi^* = \Pi^* \circ \partial_i^{\mathbb{C}}$ for $i \in \{1, ..., 2n\}$. We may also observe that for any $g \in L^1(\mathbb{CP}^n)$, $\partial_{2n+1}(\Pi^* g) = 0$ in U. Combining these facts with two applications of (11), we can see that for any $g \in L^1(\mathbb{CP}^n)$,

$$\begin{aligned}(\Delta(\Pi^* g))(\omega) &= \sum_{i=1}^{2n+1} (\partial_i(\partial_i(\Pi^* g)))(\omega) \\ &= \sum_{i=1}^{2n} (\partial_i(\partial_i(\Pi^* g)))(\omega) \\ &= \sum_{i=1}^{2n} (\Pi^*(\partial_i^{\mathbb{C}}(\partial_i^{\mathbb{C}}(g))))(\omega) \\ &= \sum_{i=1}^{2n} (\partial_i^{\mathbb{C}}(\partial_i^{\mathbb{C}}(g)))(w) \\ &= (\Delta^{\mathbb{C}} g)(w).\end{aligned} \tag{12}$$

Therefore, denoting by $(\Pi^*)^{-1}$ the inverse of $\Pi^* : L^1(\mathbb{CP}^n) \to \Pi^*(L^1(\mathbb{CP}^n))$ we have $\Delta = \Delta^{\mathbb{C}} \circ (\Pi^*)^{-1}$ on $\Pi^*(L^1(\mathbb{CP}^n))$. As eigenfunctions of Δ in $L^1(S^{2n+1})$ of eigenvalue at most $4t(t+n)$ are in $P_{2t}(S^{2n+1})$, this directly shows that $\Pi^*(R_t(\mathbb{CP}^n)) \subset P_{2t}(S^{2n+1})$, so $R_t(\mathbb{CP}^n) \subset P_t(\mathbb{CP}^n)$. To the end of showing the reverse inclusion, observe from the well-known decomposition

[1, Theorem 5.7] of the space of homogeneous polynomials of fixed degree into a direct sum of spaces of homogeneous harmonic polynomials that for any $f \in P_{2t}(S^{2n+1})$ that there exist eigenfunctions $f_i \in P_i(S^{2n+1})$ of Δ with eigenvalue $i(i+2n)$ (or zero) which are homogeneous of degree i (if nonvanishing) for $i \in \{0, ..., 2t\}$ such that $f = \sum_{i=0}^{t} f_i$. Then, we see that if $f \in \Pi^*(P_t(\mathbb{CP}^n))$, we must have $f_i \in \Pi^*(P_t(\mathbb{CP}^n))$, so each f_i is an eigenfunction of $\Delta^{\mathbb{C}} \circ (\Pi^*)^{-1}$ with eigenvalue at most $4t(t+n)$. Therefore, we have $f \in \Pi^*(R_t(\mathbb{CP}^n))$, proving that $P_t(\mathbb{CP}^n) \subset R_t(\mathbb{CP}^n)$. This completes the proof that $P_t(\mathbb{CP}^n) = R_t(\mathbb{CP}^n)$ □

5.2. **Approximate t-design curves on odd-dimensional spheres.** In this subsection, we finish the proof of the main result of this work, Theorem 1.6, to address an approximate analogue of all open ($d > 3$) settings of the problem of Ehler and Gröchenig of finding sequences of t-design curves on S^d of asymptotic order of length t^{d-1} as $t \to \infty$.

Proof of Theorem 1.6. Fix notation as in the theorem statement and consider a sequence $(Y_t)_{t=0}^{\infty}$ of t-design sets on $\mathbb{CP}^n$ of asymptotically optimal size $|Y_t| \asymp t^{2n}$, as we know exists from Proposition 5.1. With notation as in Theorem 1.8, we take $W_t := W_{Y_t}$ alongside $\delta = \delta_t := \min(1, W_t)/t$ and consider the simple ε_t-approximate t-design cycle $\gamma_t := \gamma_{Y_t}$ of length $\ell(\gamma_t) = 2\pi|Y_t| + W_t - \delta_t$ guaranteed to exist from the theorem paired with Remark 3.1 for $\varepsilon_t := W_t/(\pi|Y_t|)$. Noting from the definition of W_t in Subsection 4.2 and considering the covering radius ρ_t of Y_t, we can see that $W_t \leq (|Y_t| - 1)\rho_t/4$. Therefore, noting from Proposition 5.2 combined with Lemma 5.3 that $\rho_t \asymp 1/t$ and that $|Y_t| \asymp t^{2n}$ as $t \to \infty$, we have that $W_t \lesssim t^{2n-1}$ as $t \to \infty$. Therefore, we have $\delta_t \lesssim 1/t$ as $t \to \infty$, so $\ell(\gamma_t) \asymp t^{2n}$ and $\varepsilon_t \asymp t^{2n}/t^{2n+1} = 1/t$ as $t \to \infty$. □

5.3. **Approximate t-design curves on even-dimensional spheres.** Consider the result of Ehler and Gröchenig [12, Sections 4-6] which provides a construction of t-design curves on S^d from t-design curves on S^{d-1}, which we note extends to the setting of ε_t-approximate t-design curves:

Proposition 5.4 (Sections 4-6 of Ehler and Gröchenig [12])**.** *Fix $d \in \{2, 3, ...\}$, $\varepsilon_t \geq 0$ for $t \in \mathbb{N}$, and a sequence $(\alpha_t)_{t=0}^{\infty}$ of ε_t-approximate t-design curves on S^{d-1}. There exists a sequence $(\gamma_t)_{t=0}^{\infty}$ of ε_t-approximate t-design curves on S^d of asymptotic order $\ell(\gamma_t) \asymp t^{d-1}\ell(\alpha_t)$ of arc length as $t \to \infty$.*

This extension to ε_t-approximate t-design curves follows exactly as in the original proof, so we appeal to that work [12, Sections 4-6] to prove Proposition 5.4. Theorem 1.7 then directly follows from combining Proposition 5.4 with Theorem 1.6.

6. Further work

There are numerous avenues for continued study of weighted and approximate t-design curves. In the case of weighted curves, it would be interesting to provide explicit constructions of asymptotically optimal sequences of weighted t-design curves on S^d for all $d \geq 4$, as was done for $d \in \{2, 3\}$ in Theorem 1.3. It would also be interesting to prove existence of asymptotically optimal sequences of *simple* weighted t-design curves on S^d for $d \neq 3$. A construction analogous to that of Theorem 1.5 on complex projective spaces paired with a generalization of Theorem 2.2 we plan to formalize in future work which gives a construction that builds weighted t-design curves on odd-dimensional spheres from weighted $\lfloor t/2 \rfloor$-design curves on complex projective spaces would provide such a result. It would also be interesting to establish lower bounds on the asymptotic order of length of a sequence of weighted (and, for that matter, unweighted) t-design curves on S^d as $d \to \infty$ and to prove existence of or explicitly construct sequences achieving this bound. Finally, providing formulas for more short weighted t-design curves would also be of interest.

In the case of approximate t-design curves, it would be interesting to establish lower bounds depending on ε_t on the asymptotic order of length of a sequence of ε_t-approximate t-design curves on S^d as $t \to \infty$ or $d \to \infty$ and to investigate existence results for or constructions of sequences achieving these bounds. One subproblem of interest is investigating the existence of sequences $(\gamma_t)_{t=0}^{\infty}$ of ε_t-approximate t-design curves satisfying $\varepsilon_t \to 0$ as $t \to \infty$ of arc length $\ell(\gamma_t) \asymp t^{2n-1}$ on S^{2n} for any $n > 1$, the asymptotically optimal arc length of a sequence of (non-approximate) t-design curves on S^{2n+1}. Another avenue of inquiry is determining whether there is potential for "beautification" processes as in work of Ehler, Gröchenig, and Karner [Section 3] to be used to produce curves satisfying interesting properties (for example, (non-approximate) t-design curves) from the curves produced as in Theorem 1.8. The creation of more specific examples of approximate t-design curves for beautification and direct approximation purposes is also a potential area of interest. Combining Theorem 1.8 with constructions of t-design sets on $\mathbb{CP}^n$ [17, 20, Tables 3 and 9.2] or $S^2 \cong \mathbb{CP}^1$ [2, 16, 32] gives many such examples. Further directions of inquiry are discussed in Subsections 6.1 and 6.2.

6.1. Hybrid t-designs.

Hybrid t-design were recently introduced by Ehler to be triples (X, γ, ρ) of a finite subset $X \subset S^d$, a continuous, piecewise smooth, closed curve $\gamma : [0, 1] \to S^d$ with finitely many self-intersections, and some $\rho \in [0, 1]$ such that

$$\frac{\rho}{|X|} \sum_{x \in X} f(x) + \frac{1-\rho}{\ell(\gamma)} \int_{\gamma} f = \int_{S^d} f \, d\sigma$$

for all $f \in P_t(S^d)$ [11, Definition 4.1]. The construction of Theorem 1.8 takes inspiration from the construction of König which builds a t-design set on S^d by placing the vertices of a regular $(t+1)$-gon on the preimages of the complex projective map $\Pi : S^d \to \mathbb{CP}^n$ associated to a $\lfloor t/2 \rfloor$-design set Y on $\mathbb{CP}^n$ [18, Corollary 1]. Picking such Y alongside any $y_0 \in Y$, we take X_0 to be the union of the vertices of a regular $(t+1)$-gon on each element of $Y_{y_0} := Y \setminus \{y_0\}$ and $\gamma_0 : [0,1] \to y_0$ to be the smooth closed curve which traces around the great circle $y_0 \subset S^d$. Then, the proof of König [18, Corollary 1] (or equivalent proofs of other authors who later discussed this phenomenon [19, 21]) may be directly used to show that $(X_0, \gamma_0, 1 - 1/|Y|)$ is a hybrid t-design. We may also see that $(X_0, \gamma_0, 0)$ and $(X_0, \gamma_0, 1)$ are not hybrid t-designs for $t > 1$. In the former case, we may see this because $(X_0, \gamma_0, 0)$ being a hybrid t-design is equivalent to γ_0 being a t-design curve and this is impossible since γ_0 has image a great circle and thus can not be a 2-design curve. In the latter case, we may see this because if $(X_0, \gamma_0, 1)$ were a hybrid t-design, that would imply that $\Pi(X_0) = Y_0 = Y \setminus \{y_0\}$ were a t-design set, which is impossible since Y is a t-design set. This combined with any construction of a $\lfloor t/2 \rfloor$-design set on $\mathbb{CP}^n$ [17, 20, Tables 3 and 9.2] or $S^2 \cong \mathbb{CP}^1$ [2, 16, 32] gives a construction of a hybrid t-designs on S^d whose constituent sets and curves are not t-design sets and t-design curves respectively. Similarly, this combined with results on the asymptotic sizes of t-design sets on complex projective spaces as $t \to \infty$ [14, Theorem 2.2] shows that there exist sequences of such hybrid t-designs with non-extremal balancing constants whose constituent point sets and curves have asymptotic orders of size and length t^{d-1} and 1 respectively.

This construction is one example of a general strategy for providing straightforward examples of hybrid t-design curves communicated by Remark 6.1 which directly follows from the fact that the vertices of a regular $(t+1)$-gon on S^1 are a t-design set on S^1.

Remark 6.1. Consider $d \in \mathbb{N}_+$, $t \in \mathbb{N}$, a hybrid 0-design (X, γ, ρ) on S^d, $s > t$, and a plane $P \subset \mathbb{R}^{d+1}$ which intersects S^d in a circle such that $\gamma([0,1]) \cap P = S^d \cap P$ alongside a continuous, piecewise smooth, closed, surjective (or empty) curve $\gamma_P : [0,1] \to \gamma([0,1]) \setminus (S^d \cap P)$ with finitely many self-intersections and the vertices X_P of a regular s-gon on $S^d \cap P$. (X, γ, ρ) is a hybrid t-design if and only if $(X \cup X_P, \gamma_P, \rho + 2\pi/\ell(\gamma))$ is a hybrid t-design.

A theorem [11, Theorem 4.2(i)(iii)] of Ehler states that the triple of the set $\{(0,0,-1)\}$, the curve $s \mapsto (2\sqrt{2}\cos s, 2\sqrt{2}\sin s, 1)/3$, and the balancing constant $1/4$ constitutes a hybrid 2-design, while the set $\{(0,0,1), (0,0,-1)\}$, the curve $s \mapsto (\cos s, \sin s, 0)$, and the balancing constant $1/3$ constitutes a hybrid 3-design. In addition to being apparent from direct computation [11, Theorem 4.2(i)(iii)], these results can be verified from Remark 6.1 combined with the facts that the vertices of a tetrahedron and of an octahedron constitute 2 and 3-design sets respectively on S^2. Generalizing these results to arbitrary dimension $d \in \mathbb{N}$ using Remark 6.1, as the vertices $\{v_i\}_{i=1}^{d+1}$ of a regular simplex constitute a 2-design set on S^d, we see that the triple of the set $\{v_i\}_{i=4}^{d+1}$, the curve $s \mapsto S^d$ which traces around the unique circle containing $\{v_1, v_2, v_3\}$, and the balancing constant $1 - 3/(d+1)$ constitutes a hybrid 2-design, and since the vertices of a cross-polytope $\{\pm e_i\}_{i=1}^{d+1}$ ($\{e_i\}_{i=1}^{d+1}$ an orthonormal basis for $\mathbb{R}^{d+1}$) constitute a 3-design set in S^d, we see that the triple of the set $\{\pm e_i\}_{i=3}^{d+1}$, the curve $s \mapsto (\cos s, \sin s, 0, ..., 0)$, and the balancing constant $1 - 2/(d+1)$ is a hybrid 3-design.

6.2. **Design submanifolds.** It was noted in Subsection 4.1 that it is possible to adapt the methods of this work to produce copies of S^3 and S^7 embedded in spheres which approximately average degree at most t polynomials on spheres from t-design sets on quaternionic projective spaces, the octonionic projective lines, and certain lower-dimensional spheres. To formalize objects such as these, we introduce the notion of a (ε, c)-*approximate* P-*design submanifold* of a smooth manifold M and a subspace $P \subset L^1(M)$, which we define to be an immersed almost everywhere smooth submanifold $N \subset M$ such that

$$\left| c \int_N f \, d\omega_N - \int_M f \, d\omega_M \right| \leq \varepsilon \left| \sup_M f \right| \tag{13}$$

for all $f \in P$, where ω_N and ω_M are the volume forms on N and M respectively, normalized such that $\omega_N(N) = \omega_M(M) = 1$. When $\varepsilon = 0$, we say that N is a P-*design submanifold*. When $P_t(M)$ is a prescribed notion of polynomials on M (for example, when $P_t(M)$ is the space of *diffusion polynomials* (eigenfunctions of the Laplace-Beltrami operator) or *algebraic polynomials* (polynomials arising from an embedding of M into Euclidean space) on M), we say that N is a (ε, c)-*approximate* t-*design submanifold* of M. And finally, we may also call N a (ε, c)-*approximate* t-*design* N-*submanifold* on M, where the latter N is understood to be the homeomorphism class of N (so, for example, t-design curves would be t-design S^1-submanifolds).

It is possible to use a variety of principal bundles to produce constructions of (ε, c)-approximate t-design submanifolds. In general, applying the methods of this work to any bundle which gives rise to a result analogous to Lemma 4.1 provides such a construction.

In addition to this, constructions involving principal bundles can be used to produce (non-approximate) t-design submanifolds. One example of such a construction is the result of the present author [22, Theorem 1.3] presented in the weighted setting in Theorem 2.2 which was used to prove existence of asymptotically optimal t-design curves on S^3. As mentioned in Section 6, this construction generalizes to a construction which builds weighted t-design curves on odd-dimensional spheres from weighted $\lfloor t/2 \rfloor$-design curves

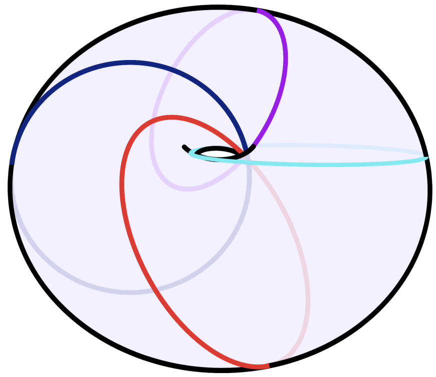

FIGURE 6. A 3-design T^2-submanifold of $S^3 \cong \mathbb{R} \cup \{\infty\}$.

on complex projective spaces which we plan to formalize in future work, but it also generalizes further to a construction capable of building a P-design submanifold of the total space M of a principal bundle $\pi : M \to N$ satisfying certain conditions from a $I_\pi P$-design submanifold A and an almost everywhere smoothly varying $P|_{\pi^{-1}(x)}$-design submanifold on $\pi^{-1}(x)$ for each $x \in A$. Future work may explore P-design submanifolds which arise from this construction.

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