ASYMPTOTICALLY OPTIMAL t-DESIGN CURVES ON S^3

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Abstract. Ehler and Gröchenig defined a *spherical t-design curve* to be a continuous, piecewise smooth, closed curve on the sphere with finitely many self-intersections whose associated line integral applied to any polynomial of degree at most t evaluates to the average of this polynomial on the sphere. These authors posed the problem of proving that there exist sequences $(\gamma_t)_{t=0}^{\infty}$ of t-design curves on S^d of asymptotically optimal length $\ell(\gamma_t) \asymp t^{d-1}$ as $t \to \infty$ and solved this problem for $d = 2$. This work solves the problem for $d = 3$ by proving that there exists a constant $\mathscr{C} > 0$ such that for any $L > \mathscr{C} t^2$ and $t \in \mathbb{N}_+$, there exists a simple t-design curve on S^3 of length L.

1. Background and main results

A finite subset $X \subset S^d$ is said to be a *spherical t-design* if we have

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^d|} \int_{S^d} f \, d\sigma$$

for all $f \in P_t(S^d)$, where $P_t(S^d)$ is the space of restrictions to S^d of polynomials on $\mathbb{R}^{d+1}$ of degree at most t and σ is the standard uniform measure on S^d. Said in different language, spherical t-designs are finite subsets of the sphere which provide equal-weight *quadrature* or *cubature* formulas for polynomials on the sphere of degree at most t. Since their introduction by Delsarte, Goethals, and Seidel [17], spherical t-designs have seen substantial interest [4, 5, 6, 7, 16, 28, 31, 44] for providing "good" discrete approximations of the sphere and for their applications and connections to combinatorics [6, 16], numerical analysis [2, 3, 9, 47], signal processing [13, 26, 49, 50], quantum information theory [1, 25, 42, 43], and beyond. Numerous important highly symmetric point configurations provide especially small t-designs [6, 17, 28, 45]; such t-designs are of interest for this reason and for the efficiency afforded by their small size in practical applications. This motivates the problem of determining the size of the smallest t-design on S^d. For $d = 1$, this size is $t + 1$, achieved by the vertices of a regular $(t + 1)$-gon on S^1 [17, Example 5.14]. However, for $d \geq 2$, even the asymptotic order of size of the smallest t-design on S^d as $t \to \infty$ remained unknown for decades. This major open problem in the field was finally resolved when Bondarenko, Radchenko, and Viazovska showed that lower bounds [17, Theorems 5.11, 5.12] on the sizes of t-designs on S^d are optimal up to a constant C_d for each d, specifically proving that there exists a spherical t-design on S^d of size N for any $t \in \mathbb{N}_+$ and $N \geq C_d t^d$ [7, Theorem 1].

Curves on spheres have applications to the collection and processing of data [11, 18, 30, 32, 36, 48] through techniques such as mobile sampling [46], a method of experimental design in which a single sensor collects data along a curve (which may be preferable to point-based sampling for reasons including cost and potential reductions to the problem of spatial aliasing [14, 46]). Curves have also been of substantial interest as the focus of natural geometric and physical optimization problems [12, 20, 21, 23, 24, 35, 40] which can be interpreted as curve-based analogues of discrete-geometric problems related to the study of t-designs [10, 16, 17, 27, 37]. Motivated by these applications and connections, Ehler and Gröchenig defined a *t-design curve* to be a continuous piecewise smooth closed curve $\gamma : [0,1] \to S^d$ with finitely many self-intersections which satisfies

$$\frac{1}{\ell(\gamma)} \int_\gamma f := \frac{1}{\ell(\gamma)} \int_0^1 f(\gamma(s))|\gamma'(s)|\, ds = \frac{1}{|S^d|} \int_{S^d} f \, d\sigma$$

for all $f \in P_t(S^d)$ [19, Definition 2.1]. These authors established an asymptotic lower bound $\ell(\gamma) \geq \widetilde{c}_d t^{d-1}$ ($\widetilde{c}_d$ a constant depending on d) on the order of length of a spherical t-design curve γ on S^d for $d \in \mathbb{N}_+$ [19, Theorem 1.1] and called sequences $(\gamma_t)_{t=0}^\infty$ of t-design curves achieving this bound up to a constant *asymptotically optimal*. They then used the results of Bondarenko, Radchenko, and Viazovska [7, Theorem 1] to show existence of asymptotically optimal sequences of t-design curves on S^d for $d = 2$ [19, Theorem 1.2] and posed the problem of proving existence of such sequences for $d > 2$ as the next question to address in the study of t-design curves [19, Section 1]. Theorem 1.1 solves this problem for $d = 3$:

Theorem 1.1 (Main Theorem)**.** *There exists a constant $\mathscr{C} > 0$ such that for any $L > \mathscr{C} t^2$ and $t \in \mathbb{N}_+$, there exists a simple t-design curve on S^3 of length L.*

We note that applying techniques of Ehler and Gröchenig [19, Sections 4-6] to a sequence $(\gamma_t)_{t=1}^\infty$ of t-design curves satisfying $\ell(\gamma_t) = \mathscr{C} t^2$, one may decrease the shortest known asymptotic order of length achieved by sequences of t-design curves on S^d for $d \geq 3$ from $t^{d(d-1)/2}$ to $t^{d(d-1)/2-1}$.

To show Theorem 1.1, we combine a construction communicated by Theorem 1.2 which builds a t-design curve on S^3 from a $\lfloor t/2 \rfloor$-design curve on S^2 with the result of Ehler and Gröchenig that there exists an asymptotically optimal sequence of t-design curves on S^2.

Theorem 1.2. *Consider $t \in \mathbb{N}$ and a $\lfloor t/2 \rfloor$-design curve β on S^2. For any $\varepsilon \geq 0$, we may construct a t-design curve $\gamma_{\beta,\varepsilon}$ on S^3 (which may have* *self-intersections for* $\varepsilon = 0$ *but which will be simple for* $\varepsilon > 0$*) of length*

$$\ell(\gamma_{\beta,\varepsilon}) = (t+1)\sqrt{\frac{\ell(\beta)^2}{4} + \phi_\beta^2} + \varepsilon \tag{1}$$

for a constant $\phi_\beta \in (-\pi, \pi]$ *(which is chosen as in* (16) *and satisfies the bound* (4)*).*

Theorem 1.3 (Theorem 1.2 of Ehler and Gröchenig [19])**.** *There exists a constant* $\mathscr{B} > 0$ *such that for any* $t \in \mathbb{N}_+$*, there exists a* t*-design curve on* S^2 *of length at most* $\mathscr{B}t$*.*

Taking $\mathscr{C} := \sqrt{\mathscr{B}^2/4 + 4\pi^2}$, Theorem 1.1 follows from Theorems 1.2 and 1.3: for any $t \in \mathbb{N}_+$ and β a $\lfloor t/2 \rfloor$-design curve satisfying $\ell(\beta) \leq \mathscr{B}\lfloor t/2 \rfloor$, we will have $\ell(\gamma_{\beta,\varepsilon}) \leq \mathscr{C}t^2$ for $\varepsilon = 0$, and we may then increase the length of $\gamma_{\beta,\varepsilon}$ as desired by increasing ε. We prove Theorem 1.2 in Section 2, then discuss applications of the construction of Theorem 1.2 and results arising from natural generalizations of the theorem in Section 3. These applications include the construction of explicit simple t-design curves on S^3 for $t \in \{1, \ldots, 27\}$, while related results include the construction of explicit sequences of asymptotically optimal t-design curves on $(S^1)^d$. We then conclude with a brief discussion of future applications of the fundamental ideas underlying Theorem 1.2.

2. Building t-design curves using the Hopf map

We now give an informal overview of the proof of Theorem 1.2, then present lemmas used in the proof in Subsection 2.1, and then provide the formal proof in Subsection 2.2. To this end, consider the Hopf map

$$\pi : S^3 \to S^2, \quad (x_1, x_2) \in \mathbb{C}^2 \mapsto (|x_1|^2 - |x_2|^2, 2x_1\overline{x}_2) \in \mathbb{R} \times \mathbb{C}, \tag{2}$$

which gives rise to a principal S^1-bundle with fibers

$$\pi^{-1}(y) = \{x\zeta \mid \zeta \in S^1 \subset \mathbb{C}\} \cong S^1 \tag{3}$$

for $y \in S^2$ and any $x \in \pi^{-1}(y)$ [29, Example 4.45]. To build the t-design curve $\gamma_{\beta,\varepsilon}$ on S^3 discussed in Theorem 1.2 from a $\lfloor t/2 \rfloor$-design curve β on S^2, we first use Lemma 2.1 to lift β to a curve $\widetilde{\beta}$ on S^3 satisfying $\pi \circ \widetilde{\beta} = \beta$ whose derivative $\widetilde{\beta}'(s)$ is always orthogonal to the tangent space of the fiber $\pi^{-1}(\pi(\widetilde{\beta}(s)))$. We then rotate $\widetilde{\beta}$ fiberwise so that the concatenation $\gamma_{\beta,\varepsilon}$ of $t+1$ appropriately rotated copies of the resulting curve is continuous, piecewise smooth, and closed and, when $\varepsilon > 0$, has no self-intersections and is lengthened by ε. We note that the lengthening of the curve done to ensure that $\gamma_{\beta,\varepsilon}$ will be closed results in the constant ϕ_β in the formula (1) for the length of $\gamma_{\beta,\varepsilon}$. We may observe from our choice (16) of ϕ_β that ϕ_β may be expressed as a function of the area enclosed by β using the Gauss-Bonnet theorem [8, 22]. Additionally, denoting by $G_{t+1} := \{g_1 < \cdots < g_T\}$ the set of generators of the cyclic group of order $t+1$ (natural numbers less than and coprime to $t+1$) and setting $g_{T+1} := g_1 + t + 1$, we can see that

$$|\phi_\beta| \leq \frac{\pi}{t+1} \max_{1\leq j\leq T} (g_{j+1} - g_j). \tag{4}$$

The construction described above will arrange that

$$(\pi \circ \gamma_{\beta,\varepsilon})(s) = \beta((t+1)s - \lfloor (t+1)s \rfloor)$$

and that

$$(\gamma_{\beta,\varepsilon}([0,1])) \cap \pi^{-1}(\pi(\gamma_{\beta,\varepsilon}(s)))$$

is the set of vertices of a regular $(t+1)$-gon on $\pi^{-1}(\pi(\gamma_{\beta,\varepsilon}(s))) \cong S^1$ for all but finitely many $s \in [0,1]$, and we may see from Lemma 2.4—the result of an example [17, Example 5.14] of Delsarte, Goethals, and Seidel—that this set of vertices is a t-design on the fiber. We show that such a curve will be a t-design curve on S^3 using a method analogous to that first used by König [33, Corollary 1] and Kuperberg [34, Theorem 4.1] to relate t-designs on spheres to $\lfloor t/2 \rfloor$-designs on quotient complex projective spaces, later used by Okuda [41, Theorem 1.1] (who was inspired by work of Cohn, Conway, Elkies, and Kumar [15, Section 4] on the subject) to relate t-designs on S^3 to $\lfloor t/2 \rfloor$-designs on S^2, and most recently used by the present author [39, Theorem 1.1] to relate t-designs on spheres to $\lfloor t/2 \rfloor$-designs on quotient real, complex, quaternionic, or octonionic projective spaces or spheres. To this end, we will present Lemmas 2.2 and 2.3, which were used by Okuda in formalizing their result [41, Theorem 1.1].

We may observe from this outline of the construction of $\gamma_{\beta,\varepsilon}$ and the presentation

$$\pi^{-1}(\xi,\eta) = \begin{cases} \left\{ \frac{1}{\sqrt{2}}\left(\zeta\sqrt{1+\xi}, \frac{\overline{\eta}\zeta}{\sqrt{1+\xi}}\right) \,\middle|\, \zeta \in S^1 \right\}, & \xi \neq -1 \\ \{(0,\zeta) \mid \zeta \in S^1\}, & \xi = -1 \end{cases}$$

of preimages of π that

$$\gamma_{\beta,\varepsilon}(s) = \begin{cases} \frac{1}{\sqrt{2}}\left(\sqrt{1+\beta_1(r)}, \frac{\beta_2(r) - i\beta_3(r)}{\sqrt{1+\beta_1(r)}}\right) e^{2\pi i g s + i\theta(r)}, & \beta_1(r) \neq -1 \\ (0,1)\, e^{2\pi i g s + i\theta(r)}, & \beta_1(r) = -1 \end{cases} \tag{5}$$

for $r := (t+1)s - \lfloor (t+1)s \rfloor$, $g \in G_{t+1}$ a generator of the cyclic group of order $t+1$, and $\theta : [0,1] \to \mathbb{R}$ a piecewise smooth function satisfying $\theta(1) - \theta(0) \in 2\pi\mathbb{Z}$ which is continuous at all $r \in [0,1]$ satisfying $\beta_1(r) \neq -1$.

2.1. **Curves and polynomials related through the Hopf map.** We now present lemmas used in the proof of Theorem 1.2. Lemma 2.1 discusses how to lift a curve on S^2 to a curve on S^3 whose derivative is orthogonal to the tangent space of each Hopf fiber it passes through, Lemma 2.2 discusses how an integrable function on S^3 can be averaged by averaging the function on S^2 which averages it on each Hopf fiber, Lemma 2.3 relates polynomials on S^3 to those on S^2 and on Hopf fibers, and Lemma 2.4 discusses how the vertices of a regular $(t+1)$-gon on a Hopf fiber give a t-design on the fiber. To show Lemma 2.1, we consider the decomposition

$$T_xS^3 \cong T_x(\pi^{-1}(y)) \oplus (T_x(\pi^{-1}(y)))^\perp \tag{6}$$

for $x \in S^3$ and $y := \pi(x) \in S^2$. The restriction

$$d\pi_x : (T_x(\pi^{-1}(y)))^\perp \to T_yS^2$$

of the differential of π at x then has inverse we denote by

$$\iota_x : T_yS^2 \to (T_x(\pi^{-1}(y)))^\perp$$

which satisfies

$$d\pi_x \circ \iota_x = \mathrm{id}_{T_yS^2}, \quad |\iota_x(v)| = \frac{1}{2}|v|, \tag{7}$$

where $\mathrm{id}_{T_yS^2}$ is the identity map on T_yS^2.

Lemma 2.1. *Take a continuous, piecewise smooth curve $\beta : [0,1] \to S^2$ with finitely many self-intersections. We may construct a continuous, piecewise smooth curve $\widetilde{\beta} : [0,1] \to S^3$ with finitely many self-intersections satisfying*

$$\widetilde{\beta}'(s) = \iota_{\widetilde{\beta}(s)}(\beta'(s)) \tag{8}$$

for all $s \in [0,1]$ at which β is smooth and

$$\pi \circ \widetilde{\beta} = \beta. \tag{9}$$

Proof. After potential reparameterization, we may assume that $|\beta'|$ is nonvanishing. Consider β as in the lemma and a partition $0 = s_0 < \cdots < s_m = 1$ of $[0,1]$ such that β has no self-intersections on each interval $I_j := [s_j, s_{j+1}]$ and is smooth on the interior of each such interval. Fixing $j \in \{0, \ldots, m-1\}$, the restriction $\beta_j := \beta|_{I_j}$ is then a diffeomorphism onto its image $\beta(I_j)$, so $\pi^{-1}(\beta(I_j))$ is a smooth submanifold (with boundary) of S^3. We define a vector field V_j on $\pi^{-1}(\beta(I_j)) \ni x$ by

$$(V_j)_x = \iota_x(\beta'(\beta_j^{-1}(\pi(x)))).$$

The existence and uniqueness theorem for ordinary differential equations allows us to construct the unique smooth flow

$$\widetilde{\beta}_j : \pi^{-1}(\beta(s_j)) \times I_j \to S^3$$

satisfying

$$\widetilde{\beta}_j(x, s_j) = x, \quad \frac{\partial \widetilde{\beta}_j}{\partial s}(x,s) = (V_j)_{\widetilde{\beta}_j(x,s)} \tag{10}$$

for $x \in \pi^{-1}(\beta(s_j))$ and $s \in I_j$. For such x, we have

$$(\pi \circ \widetilde{\beta}_j)(x, s_j) = \pi(x) = \beta(s_j)$$

and (7) shows that for any $s \in I_j$,

$$\begin{aligned}\frac{\partial(\pi \circ \widetilde{\beta}_j)}{\partial s}(x,s) &= \left(d\pi_{\widetilde{\beta}_j(x,s)} \circ \frac{\partial \widetilde{\beta}_j}{\partial s}\right)(x,s) \\ &= d\pi_{\widetilde{\beta}_j(x,s)}((V_j)_{\widetilde{\beta}_j(x,s)}) \\ &= \beta'(\beta_j^{-1}((\pi \circ \widetilde{\beta}_j)(x,s))).\end{aligned}$$

On the other hand, β_j solves the same ordinary differential equation. By uniqueness of such solutions, for $x \in \pi^{-1}(\beta(s_j))$ and $s \in I_j$, we thus have

$$(\pi \circ \widetilde{\beta}_j)(x,s) = \beta(s). \tag{11}$$

We note this shows that $\widetilde{\beta}_j$ has no self-intersections since β_j has none. Now, fix $x \in \pi^{-1}(\beta(0))$. We consider the curve $\widetilde{\beta} : [0,1] \to S^3$ which arises from flowing along each $\widetilde{\beta}_j$ successively starting at x, so we have $\widetilde{\beta}(0) = x$ and $\widetilde{\beta}(s) = \widetilde{\beta}_j(\widetilde{\beta}(s_j), s)$ for $j \in \{0, \ldots, m-1\}$ and $s \in I_j$. We see since $\widetilde{\beta}_j$ is smooth and has no self-intersections for each j that $\widetilde{\beta}$ is continuous and piecewise regular with finitely many self-intersections. Moreover, we directly see from (10) and (11) that (8) and (9) hold. □

Lemmas 2.2 and 2.3 are concerned with properties of the operator I_π which takes $f \in L^1(S^3)$ to the function

$$(I_\pi f)(y) = \frac{1}{2\pi} \int_{\pi^{-1}(y)} f \, d\sigma \tag{12}$$

on S^2 which gives the average of f on each Hopf fiber (where σ is the standard uniform measure on $\pi^{-1}(y) \cong S^1$) and the pullback $\zeta_x^* : f \mapsto f \circ \zeta_x$ of the isomorphism

$$\zeta_x : S^1 \to \pi^{-1}(y), \quad \zeta \mapsto x\zeta \tag{13}$$

we consider for $x \in S^3$ and $y := \pi(x) \in S^2$. We do not prove these lemmas, but note that proofs can be found in work of Okuda [41, Lemmas 4.2-4.3] or of the present author [39, Lemmas 2.2-2.3, 2.5].

Lemma 2.2 (Lemma 4.2 of Okuda [41])**.** *For $f \in L^1(S^3)$, we have*

$$\frac{1}{|S^2|} \int_{S^2} I_\pi f \, d\sigma = \frac{1}{|S^3|} \int_{S^3} f \, d\sigma.$$

Lemma 2.3 (Lemma 4.3 of Okuda [41])**.** *We have*

$$I_\pi(P_t(S^3)) = P_{\lfloor t/2 \rfloor}(S^2), \tag{14}$$

$$\zeta_x^*(P_t(S^3)) = P_t(S^1) \quad \text{for any} \quad x \in S^3. \tag{15}$$

Lemma 2.4 follows from (15) paired with the fact [17, Example 5.14] noted by Delsarte, Goethals, and Seidel that the vertices of a regular $(t+1)$-gon give a t-design on S^1.

Lemma 2.4 (Example 5.14 of Delsarte, Goethals, and Seidel [17])**.** *For any $x \in S^3$ and $f \in P_t(S^3)$, we have*

$$\frac{1}{t+1}\sum_{j=0}^{t} f(xe^{2\pi ij/(t+1)}) = (I_\pi f)(\pi(x)).$$

2.2. **Formal treatment of the construction.** We now prove Theorem 1.2, applying Lemma 2.1 to assist in constructing the desired curve $\gamma_{\beta,\varepsilon}$ and using Lemmas 2.2, 2.3, and 2.4 to show that $\gamma_{\beta,\varepsilon}$ is a t-design curve.

Proof of Theorem 1.2. Consider β as in the theorem statement, which we reparameterize so its derivative has constant norm $|\beta'| = \ell(\beta)$. Consider $\widetilde{\beta}$ as in Lemma 2.1 alongside the generators G_{t+1} of the cyclic group of order $t+1$. Since β is a closed curve, (9) shows that $\widetilde{\beta}(0)$ lies in the image $\pi^{-1}(\pi(\widetilde{\beta}(1)))$ of the isomorphism $\zeta_{\widetilde{\beta}(1)}$ as in (13). Therefore, we may pick $g \in G_{t+1}$ minimizing $|\phi_\beta|$ for $\phi_\beta \in (-\pi, \pi]$ defined by

$$e^{i\phi_\beta} = \zeta_{\widetilde{\beta}(1)}^{-1}(\widetilde{\beta}(0))e^{2\pi ig/(t+1)} \in S^1 \subset \mathbb{C}. \tag{16}$$

With ε as in the theorem statement, fix

$$\phi_\varepsilon := \sqrt{\phi_\beta^2 + \frac{2\varepsilon}{t+1}\sqrt{\frac{\ell(\beta)^2}{4} + \phi_\beta^2} + \frac{\varepsilon^2}{(t+1)^2}},$$

so we have

$$\sqrt{\frac{\ell(\beta)^2}{4} + \phi_\varepsilon^2} = \sqrt{\frac{\ell(\beta)^2}{4} + \phi_\beta^2} + \frac{\varepsilon}{t+1}. \tag{17}$$

We may consider the partition $0 = s_0 < \cdots < s_m = 1$ of $[0,1]$ arising from the union of $\{0,1\}$ with the preimage under β of the self-intersection points of β (and with an arbitrary extra point of $(0,1)$ if β is simple, so we have $m > 1$). Writing

$$r_{j,\delta} := \frac{1}{2}(s_{j-1} + s_j + (s_j - s_{j-1})\phi_\beta/\phi_\varepsilon) + \delta \quad \text{for} \quad j \in \{1, \ldots, m-1\},$$
$$r_{m,\delta} := \frac{1}{2}(s_{m-1} + 1 + (1 - s_{m-1})\phi_\beta/\phi_\varepsilon) - (m-1)\delta$$

for $\delta \geq 0$, we set

$$\Delta = \min\left\{s_1 - r_{1,0}, \ldots, s_{m-1} - r_{m-1,0}, \frac{r_{m,0} - s_{m-1}}{m-1}\right\}$$

and fix $\delta \in [0, \Delta]$, so we have $r_{j,\delta} \in [s_{j-1}, s_j]$ for all j. Setting

$$I_{+,\delta} := \bigcup_{j=1}^{m}(s_{j-1}, r_{j,\delta}), \quad I_{-,\delta} := \bigcup_{j=1}^{m}(r_{j,\delta}, s_j),$$

we can directly compute that

$$|[0,s_j]\cap I_{+,\delta}|-|[0,s_j]\cap I_{-,\delta}| = s_j\phi_\beta/\phi_\varepsilon+2j\delta \quad \text{for} \quad j\in\{1,\dots,m-1\},$$
$$|I_{+,\delta}|-|I_{-,\delta}| = \phi_\beta/\phi_\varepsilon.$$

Thus, considering the unique continuous function $\theta_\delta:[0,1]\to\mathbb{R}$ defined by

$$\theta_\delta(0)=0, \quad \theta_\delta'(s)=\pm\phi_\varepsilon \quad \text{for} \quad s\in I_{\pm,\delta},$$

we have that

$$\begin{aligned} \theta_\delta(s_j) &= s_j\phi_\beta+2j\delta\phi_\varepsilon \quad \text{for} \quad j\in\{0,\dots,m-1\},\\ \theta_\delta(1) &= (|I_{+,\delta}|-|I_{-,\delta}|)\phi_\varepsilon=\phi_\beta. \end{aligned} \tag{18}$$

Defining functions

$$q(s)=\frac{\lfloor (t+1)s\rfloor}{t+1}, \quad r(s)=(t+1)(s-q(s))$$

for $s\in[0,1]$, we consider the continuous, piecewise smooth curve

$$\widetilde{\gamma}_{\beta,\delta}:=(\widetilde{\beta}\circ r)e^{i(\theta_\delta\circ r+2\pi gq)}:[0,1]\to S^3.$$

We may directly see that

$$\widetilde{\gamma}_{\beta,\delta}(1)=\widetilde{\beta}(0)=\widetilde{\gamma}_{\beta,\delta}(0),$$

so $\widetilde{\gamma}_{\beta,\delta}$ is a closed curve. Observe from (9) that

$$\pi\circ\widetilde{\gamma}_{\beta,\delta}=\pi\circ\widetilde{\beta}\circ r=\beta\circ r,$$

so $\widetilde{\gamma}_{\beta,\delta}$ may only have a self-intersection at a point $s\in[0,1]$ if there exists $\tilde{s}\in[0,1]$ such that $r(s)=r(\tilde{s})$ or if $r(s)$ is a self-intersection point of β. If $r(s)=r(\tilde{s})$, we have $\tilde{s}=s+k/(t+1)$ for some $k\in\{-t,\dots,-1,1,\dots,t\}$, so

$$\widetilde{\gamma}_{\beta,\delta}(\tilde{s})=\widetilde{\beta}(r(s))e^{i(\theta_\delta(r(s))+2\pi g(q(s)+k/(t+1)))}=\widetilde{\gamma}_{\beta,\delta}(s)e^{2\pi igk/(t+1)}\neq\widetilde{\gamma}_{\beta,\delta}(s)$$

since g is a generator of the cyclic group of $t+1$ elements and thus gk will not be an integer multiple of $t+1$. Therefore, $\widetilde{\gamma}_{\beta,\delta}$ may only have self-intersections in

$$r^{-1}(\{s_j\}_{j=0}^m)=\left\{s_{j,k}:=\frac{s_j+k}{t+1}\;\middle|\; j\in\{0,\dots,m\},\ k\in\{0,\dots,t\}\right\}.$$

For any $k\in\{0,\dots,t\}$, (18) and (16) show that

$$\widetilde{\gamma}_{\beta,\delta}(s_{j,k})=\widetilde{\beta}(s_j)e^{i(s_j\phi_\beta+2j\delta\phi_\varepsilon+2\pi gk/(t+1))} \quad \text{for} \quad j\in\{0,\dots,m-1\},$$
$$\widetilde{\gamma}_{\beta,\delta}(s_{m,k})=\widetilde{\beta}(1)e^{i(\phi_\beta+2\pi gk/(t+1))}=\widetilde{\beta}(0)e^{2\pi ig(k+1)/(t+1)},$$

so we can see that there will only be finitely many $\delta\in[0,\Delta]$ such that $\widetilde{\gamma}_{\beta,\delta}$ will have any self-intersections. We take $\gamma_{\beta,\varepsilon}:=\widetilde{\gamma}_{\beta,0}$ when $\varepsilon=0$. Otherwise, we have $\Delta>0$, so we may pick $\delta\in[0,\Delta]$ such that $\widetilde{\gamma}_{\beta,\delta}$ has no self-intersections and label $\gamma_{\beta,\varepsilon}:=\widetilde{\gamma}_{\beta,\delta}$.

Combining (8), (7), and our assumption that $|\beta'|$ is constant, we see that $|\widetilde{\beta}'| = |\beta'|/2 = \ell(\beta)/2$, so (6) shows that

$$|\gamma_{\beta,\varepsilon}'|^2 = |(\widetilde{\beta} \circ r)'|^2 + |(\theta_\delta \circ r)'|^2 = (t+1)^2 \left(\frac{\ell(\beta)^2}{4} + \phi_\varepsilon^2 \right) \tag{19}$$

at all points where $\gamma_{\beta,\varepsilon}$ is smooth. Thus, we have

$$\ell(\gamma_{\beta,\varepsilon}) = (t+1)\sqrt{\frac{\ell(\beta)^2}{4} + \phi_\varepsilon^2}$$

and (17) then shows that (1) is satisfied.

To complete the proof, we need only to show that for any $f \in P_t(S^3)$,

$$\frac{1}{\ell(\gamma_{\beta,\varepsilon})} \int_{\gamma_{\beta,\varepsilon}} f = \frac{1}{|S^3|} \int_{S^3} f \, d\sigma. \tag{20}$$

Picking such f, as (19) shows that $|\gamma_{\beta,\varepsilon}'|$ is constant and thus equals $\ell(\gamma_{\beta,\varepsilon})$, we have

$$\frac{1}{\ell(\gamma_{\beta,\varepsilon})} \int_{\gamma_{\beta,\varepsilon}} f = \frac{1}{\ell(\gamma_{\beta,\varepsilon})} \int_0^1 f(\gamma_{\beta,\varepsilon}(s)) \ell(\gamma_{\beta,\varepsilon}) \, ds = \int_0^1 f(\gamma_{\beta,\varepsilon}(s)) \, ds. \tag{21}$$

With I_π as in (12), we then see from Lemma 2.4 and since g is a generator of the cyclic group of order $t+1$ that

$$\frac{1}{t+1} \sum_{k=0}^{t} f(x e^{2\pi i g k/(t+1)}) = (I_\pi f)(\pi(x)),$$

so applying a change of variables $s \mapsto \frac{s}{t+1}$, (9), and that $|\beta'| = \ell(\beta)$, we have

$$\begin{aligned} \int_0^1 f(\gamma_{\beta,\varepsilon}(s)) \, ds &= \sum_{k=0}^{t} \int_{k/(t+1)}^{(k+1)/(t+1)} f(\gamma_{\beta,\varepsilon}(s)) \, ds \\ &= \int_0^1 \frac{1}{t+1} \sum_{k=0}^{t} f(\widetilde{\beta}(s) e^{i(\theta_\delta(s) + 2\pi g k/(t+1))}) \, ds \\ &= \int_0^1 (I_\pi f)((\pi \circ \widetilde{\beta})(s)) \, ds \\ &= \frac{1}{\ell(\beta)} \int_\beta I_\pi f. \end{aligned} \tag{22}$$

We see from (14) in Lemma 2.3 that $I_\pi f \in P_{\lfloor t/2 \rfloor}(S^2)$, so since β is a $\lfloor t/2 \rfloor$-design, (21) and (22) combine to show that

$$\frac{1}{\ell(\gamma_{\beta,\varepsilon})} \int_{\gamma_{\beta,\varepsilon}} f = \frac{1}{|S^2|} \int_{S^2} I_\pi f \, d\sigma.$$

Lemma 2.2 then shows that (20) is satisfied. $\square$

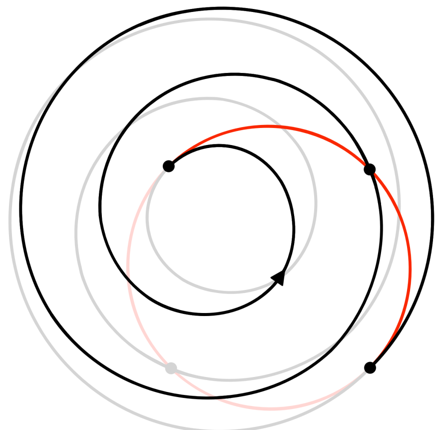

Figure 1. The 3-design curve $\gamma_{\beta,0}$ on $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ as in (23) for β the curve which traces around the equator in S^2. $\gamma_{\beta,0}$ lies on the Clifford torus that is the preimage of the equator of S^2 under the Hopf map; the segment of the curve on the same side of this torus as the observer is shown in black and the further segment is shown in gray. The preimage $\pi^{-1}(p)$ of a point on the equator of S^2 under the Hopf map is shown in red. Note that $\gamma_{\beta,0}([0,1]) \cap \pi^{-1}(p)$ is the set of vertices of a square, a 3-design on $\pi^{-1}(p)$.

3. Explicit t-design curves and generalizations

We now discuss examples of explicit t-design curves arising from the construction of Theorem 1.2. To this end, observe that the curve $\beta(s) = (0, \cos(2\pi s), \sin(2\pi s))$ which traces around the equator of S^2 is a 1-design curve on S^2. Taking $t \in \{2, 3\}$ and building $\gamma_{\beta,0}$ as in Theorem 1.2, we produce a smooth, simple t-design curve

$$(23) \quad [0,1] \to S^3, \quad s \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi s),\, \sin(2\pi s),\, \cos(2\pi t s)\,, -\sin(2\pi t s))$$

on S^3 which we may observe has length $\pi\sqrt{2t^2+2}$. This curve for $t = 3$ is pictured in Figure 1.

We can similarly use the construction of Theorem 1.2 to produce smooth, simple t-design curves on S^3 for $t \in \{4, 5, 6, 7\}$ from the smooth, simple 2- and 3-design curves discovered by Ehler and Gröchenig [19, Example 3.3]. For $t \in \{0, \ldots, 27\}$, we can produce further explicit examples of $\lfloor t/2 \rfloor$-design curves on S^2 by applying the construction of Ehler and Gröchenig [19, Section 5]—which builds a $\lfloor t/2 \rfloor$-design curve on S^2 from a $\lfloor t/2 \rfloor$-design on S^2—to the $\lfloor t/2 \rfloor$-designs on S^2 compiled by Hardin and Sloane [28, Table I]. Applying the construction of Theorem 1.2 to these curves then gives numerous additional examples of simple t-design curves on S^3.

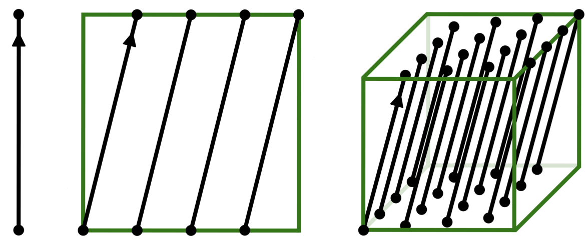

FIGURE 2. 3-design curves on S^1, $S^1 \times S^1$, and $S^1 \times S^1 \times S^1$ as in (24). For a $\lfloor t/2 \rfloor$-design curve β on S^2, $\gamma_{\beta,0}$ as in Theorem 1.2 can be realized as the composition of such a t-design curve on $S^1 \times S^1$ with a map to $\pi^{-1}(\beta([0,1]))$.

We now discuss results related to Theorem 1.2 concerning t-design curves on spaces other than spheres. Define, for any $t \in \mathbb{N}$ and $n \in \mathbb{N}_+$, a t-design curve on a measure space $(\Sigma \subset \mathbb{R}^n, \mu)$ to be a continuous, piecewise smooth, closed curve $\gamma : [0,1] \to \Sigma$ with finitely many self-intersections whose associated line integral exactly averages the restrictions of elements of $P_t(\mathbb{R}^n)$ to Σ as in the definition of spherical t-design curves presented in Section 1. Consider a t-design curve β on such (Σ, μ). The curve

$$[0,1] \to \Sigma \times S^1, \quad s \mapsto (\beta((t+1)s - \lfloor (t+1)s \rfloor), e^{2\pi i s})$$

can be shown exactly as in the proof of Theorem 1.2 (substituting for the Hopf map the projection $\Sigma \times S^1 \to \Sigma$) to be a t-design curve on $(\Sigma \times S^1 \subset \mathbb{R}^{n+2}, \mu \times \sigma)$. As the curve $\alpha : [0,1] \to S^1$ taking $s \mapsto e^{2\pi i s}$ is trivially a t-design curve on S^1 for any $t \in \mathbb{N}_+$, we see that

$$\gamma_t : [0,1] \to S^1 \times S^1, \quad s \mapsto (e^{2\pi i (t+1) s}, e^{2\pi i s})$$

is a t-design curve on the Clifford torus $S^1 \times S^1$ of length

$$\ell(\gamma_t) = 2\pi\sqrt{(t+1)^2 + 1} \leq 2\pi\sqrt{5}t.$$

We then see that for any $d \in \mathbb{N}_+$,

$$\gamma_{t,d} : [0,1] \to (S^1)^d, \quad s \mapsto (e^{2\pi i (t+1)^{d-1} s}, e^{2\pi i (t+1)^{d-2} s}, \ldots, e^{2\pi i s}) \tag{24}$$

is a t-design curve on $(S^1)^d$ of length

$$\ell(\gamma_{t,d}) = 2\pi\sqrt{\sum_{j=0}^{d-1}(t+1)^{2j}} \leq \mathscr{C}_{(S^1)^d} t^{d-1} \quad \text{for} \quad \mathscr{C}_{(S^1)^d} := 2\pi\sqrt{\sum_{j=0}^{d-1} 4^j}.$$

When $d > 1$, the curves (24) may be lengthened by an arbitrary amount as in the proof of Theorem 1.2. For any $L \geq \mathscr{C}_{(S^1)^d} t^{d-1}$ and any $t \in \mathbb{N}_+$, we may therefore construct a smooth, simple t-design curve on $(S^1)^d$ of length L. We then have, using Theorem 1.3 when $n = 2$ and Theorem 1.1 when $n = 3$, that there exists a constant $\mathscr{C}_{S^n \times (S^1)^d}$ such that for any $L > \mathscr{C}_{S^n \times (S^1)^d} t^{n-1+d}$ and $t \in \mathbb{N}_+$, there exists a (simple, for $n = 3$) t-design curve on $S^n \times (S^1)^d$

of length L. A proof of Ehler and Gröchenig [19, Theorem 1.1] can then be directly generalized to show that all of these curves achieve the minimal possible asymptotic order of length among such curves as $t \to \infty$.

Now, consider $n \in \mathbb{N}_+$ alongside the complex projective space

$$\mathbb{CP}^n := \{[x] \mid x \in S^{2n+1} \subset \mathbb{C}^{n+1}\} \quad ([x] := \{x\zeta \mid \zeta \in S^1 \subset \mathbb{C}\}).$$

We may realize $\mathbb{CP}^n$ as a subset of $\mathbb{C}^{(n+1)^2} \cong \mathbb{R}^{2(n+1)^2}$ via the embedding taking an element $[x] \in \mathbb{CP}^n$ to the matrix xx^* which projects $\mathbb{C}^{n+1}$ to the span over $\mathbb{R}$ of $[x]$, providing a notion of t-design curves on $\mathbb{CP}^n$. The Hopf map can then be realized, via the association $\mathbb{CP}^1 \cong S^2$, as the $n = 1$ instance of the complex projective map $S^{2n+1} \ni x \mapsto [x] \in \mathbb{CP}^n$. We note that the fundamental ideas underlying Theorem 1.2 and the further constructions discussed in this section reflect a broadly applicable geometric mechanism. This suggests generalizations of Theorem 1.2 (whose investigation we defer to future work [38]) which use geometric maps such as the complex projective map to provide new constructions of t-design curves. Such constructions may then be useful in providing improved bounds on the asymptotic order of length (as $t \to \infty$) of sequences of t-design curves on S^d for $d > 3$.

Acknowledgements

The author would like to thank Henry Cohn for engaging in interesting conversations about problems related to the ones addressed by this manuscript. The author would also like to thank the School of Science, the Department of Mathematics, and the Office of Graduate Education for support through the MIT Dean of Science fellowship during their doctoral studies, as well as NSF grant DMS-2105512 and the Simons Foundation Award #994330 (Simons Collaboration on New Structures in Low-Dimensional Topology).

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